"Confined Coherence" in Strongly Correlated, Anisotropic Metals

David G. Clarke
IRC in Superconductivity and Cavendish Laboratory, University of Cambridge, Cambridge, CB3 0HE, United Kingdom

S. P. Strong
NEC Research Institute, 4 Independence Way, Princeton, NJ, 08540, U.S.A.
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Abstract

We present a detailed discussion of both theoretical and experimental evidence in favour of the existence of states of "confined coherence" in metals of sufficiently high anisotropy and with sufficiently strong correlations. The defining property of such a state is that single electron coherence is confined to lower dimensional subspaces (planes or chains) so that it is impossible to observe interference effects between histories which involve electrons moving between these subspaces. The most dramatic experimental manifestation of such a state is the coexistence of incoherent, non-metallic transport in one or two directions (transverse to the lower dimensional subspaces) with coherent transport in at least one other direction (within the subspaces). The magnitude of the Fermi surface warping due to transverse (inter-subspace) momentum plays the role of an order parameter (in a state of confined coherence, this order parameter vanishes) and the effect can occur in a pure system at zero temperture. Our theoretical approach is to treat an anisotropic 2D (3D) electronic system as a collection of 1D (2D) electron liquids coupled by weak interliquid single particle hopping. We find that a necessary condition for the destruction of coherent interliquid transport is that the intraliquid state be a non-Fermi liquid. We present a very detailed discussion of coupled 1D Luttinger liquids and the reasons for believing in the existence of a phase of confined coherence in that model. This provides a paradigm for incoherent transport between weakly coupled 2D non-Fermi liquids, the case relevant to the experiments of which we are aware. Specifically, anomalous transport data in the (normal state of the) cuprate superconductors and in the low temperature, metallic state of the highly anisotropic organic conductor $(TMTSF)_2PF_6$ cannot be understood within a Fermi liquid framework, and, we argue, the only plausible way to understand that transport is in terms of

a state of confined coherence.

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I. INTRODUCTION

A. Preamble

Motivated primarily by the difficulties faced in understanding the normal state of the high-temperature superconducting cuprates (HTSC's) within the framework of Fermi liquid theory (FLT), there has been a great deal of theoretical interest in the possible existence of "metallic" states of matter controlled by non-Fermi liquid (NFL) fixed points in two, and possibly even three, dimensions. As has been known for many years, FLT is never an appropriate description of a one dimensional (1D) metal; rather, the generic paradigm to use is that of the Luttinger liquid [1], a quite specific type of NFL. In two and three dimensions it is widely accepted that, at least for sufficiently weak electron-electron interactions and generic Fermi surface shapes, the free Fermi gas is a stable fixed point. This belief is based on the stability of generic Fermi surfaces in the conventional diagrammatic perturbation theory [2] and the renormalization group [3] to everything except the usual BCS instability. However, in two dimensions there are some unresolved questions about the applicability of the starting point [4,5] and non-Fermi liquid behavior for weak interactions is somewhat controversial. Certainly, for strong interactions and/or Fermi surfaces with strong nesting properties the "NFL" spin and charge density wave states represent alternative, strong coupling fixed points, and there is a great deal of interest at present in the possible existence of other strong coupling fixed points, especially in two dimensions.

Whatever the case for 1D and 2D models, bulk materials are never purely 1D or 2D systems, *e.g.* the so-called quasi-1D organic conductors have interchain hopping integrals typically of order a few hundred degrees, and one might be led to the view that any metallic state existing at sufficiently low temperature will be three dimensional and therefore a Fermi liquid, provided that the interactions are not too strong. There are two commonly accepted ways around this argument, one being that sufficiently strong interaction can render the hopping between chains irrelevant, or at least less relevant than other interchains couplings, the other being that for Fermi surfaces with special shapes the interactions are never weak. The former is not known to occur in any actual material, while the latter is not uncommon and results in the known charge and spin density wave materials. The purpose of this paper is to review a third proposal [6–9] for avoiding the low temperature crossover to a three dimensional Fermi liquid.

Our motivation for doing this is quite simple: we are interested in trying to explain experiments we believe are incompatible with either Fermi liquid behavior or existing non-Fermi liquid proposals. The problematic experiments are those on the high T_c superconductors at low or optimal doping and those on the organic conductor $(TMTSF)_2PF_6$ in its low temperature, high pressure, metallic phase.

B. Experimental Motivation

The theoretical work presented in this paper was originally motivated by the bizarre transport properties of the HTSC's. In particular, the high T_c compounds generically display *qualitatively* anisotropic transport, which, as we will argue, is not explicable within

Fermi liquid theory. At the same time, as we will discuss, the measured experimental properties of, for example, $Bi_2Sr_2CaCu_2O_8$, rule out the possibility that the anisotropy arises from the renormalization group irrelevance of the interplane hopping. We believe there is therefore a clear and serious need for a theoretical proposal capable of producing qualitatively anisotropic transport in the face of the renormalization group relevance of the interplane hopping. Providing such a proposal is the intent of this paper.

Let us begin with a discussion of the evidence of non-Fermi liquid behavior in the cuprates. Apart from the extraordinarily high superconducting transition temperatures, T_c, these materials exhibit a highly anomalous metallic state at temperatures above T_c. Despite attempts to do so, we believe that no plausible scenario has been presented to understand this metallic state within the (Landau) Fermi liquid (FL) paradigm [10]. There are simply too many anomalies to account for. The cuprates therefore present us with a much richer problem than high T_c alone, *viz.* the existence of an essentially two-dimensional (2D) non-Fermi liquid (NFL) metallic state. The problem of understanding why the T_c's of the cuprates can be so high is intimately connected then with the problem of understanding the physics of the anomalous normal state.

For our purposes it suffices to illustrate the anomalous nature of the normal state by considering the simplest transport properties. The basic structure of all of the HTSC's is the presence of "stacked" copper-oxide (CuO_2) planes. The standard nomenclature is to refer to in-plane properties as "ab-plane", and inter-plane as "c-axis". We shall begin by considering ab-plane transport. It has long been known that the normal state resistivity $\rho(T)$ does not follow the Fermi liquid form. Empirically, the normal state resistivity can be approximately characterized by

$$\rho_{ab}(T) = a + bT^p \tag{1}$$

In the vicinity of "optimal doping", ρ_{ab} exhibits the famous "linear-T dependence", $\rho_{ab}(T) \propto T$. In contrast, for $T \lesssim 0.2\ \Theta_D \ll E_F$ the resistivity in a FL is

$$\rho_{ab}^{\rm FL}(T) \sim \rho_{\rm imp} + c_{\rm el-el}T^2 + c_{\rm el-ph}T^5 \tag{2}$$

The three terms represent, respectively, the dissipation of electrical current by scattering from impurities (temperature independent), other electrons ($\sim T^2$) and phonons ($\sim T^5$). The Debye temperature Θ_D of the HTSC's is estimated from specific heat measurements to be in the range of 300-450 K. Although the temperature range of the normal state is bounded below by T_c, the crucial observation is that there are examples of HTSC's with T_c well below $0.2\ \Theta_D$ which clearly fail to fit the FL form (2).

A second anomalous ab-plane transport property involves the Hall angle, $\theta_{\rm H} = \cot^{-1}(\sigma_{xx}/\sigma_{xy})$. As first pointed out by Chien, Wang and Ong [11], and by Anderson [12], there is a remarkably clean quadratic temperature dependence of $\cot\theta_{\rm H}$ in the normal state of the cuprates

$$\cot\theta_{\rm H} = A + BT^2 \tag{3}$$

A *Hall relaxation rate* $\tau_{\rm H}^{-1}$ can be defined via

$$\frac{1}{\tau_{\rm H}} = \omega_c \cot\theta_{\rm H} \tag{4}$$

Assuming only weak, or no, temperature dependence of ω_c this relaxation rate is observed to be distinct from the transport rate $\tau_{\rm tr}^{-1}$ derived from $\rho(T)$ and the frequency dependent in-plane conductivity [13].

Remarkably, the T^2 behavior of $\tau_{\rm H}^{-1}$ appears to be very robust to changes in doping, *i.e.* it is not restricted to "near optimal" doping [14]. Indeed, it has been observed even in very under- and over-doped samples where $\rho(T)$ is far from linear. Any successful theory of these two relaxation rates must be able to account for this apparent empirical fact that $\tau_{\rm H}$ is more robust than $\tau_{\rm tr}$.

The existence of two such fundamentally different in-plane transport relaxation rates is incompatible with conventional scattering mechanisms of Landau quasiparticles, both quasi-elastic (impurities and phonons) and inelastic (electron-electron). It is very difficult to see how other inelastic scattering mechanisms (spin fluctuations for example) could be introduced within a FL framework and be capable of reproducing these two rates [15]. There are two problems to overcome, the first to construct a physical mechanism for two relaxation rates *per se*, the second to be able to obtain the required $\tau_{\rm tr}$ without destroying the quasiparticles. If one extracts the frequency dependence of $\tau_{\rm tr}$ from ab-plane conductivity, and interprets this as the frequency dependence of the lifetime of a quasiparticle (via $Im\Sigma(\omega) \propto \tau_{\rm tr}(\omega)$), then the quasiparticle renormalization $Z_k = (1 - \partial Re\Sigma_k(\omega)/\partial\omega)^{-1}_{\omega=E_k}$ is found to vanish as $k \to k_F$, *i.e.* there is no quasiparticle at all [16,17]! This type of argument is clearly incompatible with a FL picture.

But this is not the end of the transport story. We have so far discussed only the ab-plane transport. However the c-axis transport is observed to be *qualitatively different*, the importance of which was emphasized by Anderson in the very early days of high-T_c [18]. A more detailed discussion is given in section VII , but the key concepts involved are as follows.

In La_2CuO_4, the prototypical HTSC, band theory calculations predict a single, half-filled $d_{x^2-y^2}$ band crossing the Fermi surface [19]. The ratio of the c-axis bandwidth $4t_\perp$ to the ab-plane bandwidth $4t_\parallel$ is calculated to be $t_\perp/t_\parallel \approx 1/10$. Although the band theories incorrectly predict a metallic state for La_2CuO_4, due to their inability to adequately account for strong correlation effects, it is perfectly correct to use, in a starting point ("bare") Hamiltonian, the anisotropies predicted by these methods. Upon doping to $La_{2-x}Sr_xCuO_4$ the anisotropy in the hopping should not greatly change. Ignoring any correlation effects beyond those incorporated into a FL framework, and within the isotropic relaxation time approximation, the ratio of c-axis to ab-plane resistivity is predicted to be temperature independent and of the order $\rho_c(T)/\rho_{ab}(T) \approx 25$. The experimental fact is that in all superconducting samples $\rho_c(T)/\rho_{ab}(T)$ is strongly temperature dependent, $\rho_c(T)$ actually turning upwards ($\partial\rho_c(T)/\partial T < 0$) at low temperatures in the underdoped samples, while $\rho_{ab}(T)$ remains metallic ($\partial\rho_{ab}(T)/\partial T > 0$ and ρ_{ab} well below the Mott-Ioffe-Regel limit). The anisotropy can reach a magnitude of several hundred as $T \to T_c^+$ [20]. Indeed at optimal doping the projected zero temperature ratio can be as high as several thousand or more! Moreover, even as $T \to T_c^+$ the c-axis mean free path estimated from $\rho_c(T)$ is much shorter than the interplanar spacing, thus precluding metallic Bloch-like conductivity. Equivalently, the c-axis conductivity is well below the Mott-Ioffe-Regel limit. This is a highly nontrivial result, for the interplane hopping rate is estimated from band theory to be $t_\perp \sim 500K$, yet the in-plane scattering rate is of order $\tau_{\rm tr}^{-1} \sim T$. Thus $t_\perp \gg \tau_{\rm tr}^{-1}$ over much of the experimental

temperature range, and in particular in the vicinity of $T_c \lesssim 40K$. At low temperature, and in the absence of a significant renormalization of $t_\perp$, one would therefore expect to be in a region of coherent c-axis conduction, yet the experimental evidence is strongly contrary to this. Even more compelling perhaps than the studies of the dc conductivity are the recent experimental studies of the frequency dependent c-axis conductivity $\sigma_c(\omega)$ [21–24]. There is simply no plausible way to fit the low frequency data to a Drude, or generalized Drude, form: the data are simply incompatible with a zero-frequency Drude peak with any appreciable weight and reasonable width. The analysis of $\rho_c(T)$ for bilayer and trilayer cuprates requires more care than for the single-layer compounds like $La_{2-x}Sr_xCuO_4$, but the general fact of incoherence remains. Moreover, photoemission experiments on $Bi_2Sr_2CaCu_2O_8$, a bilayer HTSC in which the inter-bilayer coupling is very weak, point to the existence of a *single* sharp Fermi surface [25]. The experimental resolution is $\lesssim 20\ meV$. This strongly contradicts the band theory prediction of *two* Fermi surfaces split by an energy of a few hundred meV. Such a splitting would be the direct consequence of coherent single electron hopping between the two CuO_2 planes within a bilayer. The absence of any splitting demonstrates that, even within a bilayer where we expect $t_\perp \gg \tau_{\rm tr}^{-1}$, interplanar hopping is incoherent.

The empirical fact that the c-axis conduction is incoherent is therefore another very strong argument against a FL scenario, as pointed out long ago by Anderson [26]. Indeed, it was the strange behavior of $\rho_c(T)$ which first led Anderson to propose "confinement" of electrons to the CuO_2 planes as the driving force behind high-T_c itself [27]. The idea we will advocate in this paper is that it is not the electrons, but the *coherence* that is confined to the *ab* planes.

The picture that therefore emerges is that of a macroscopic number of 2D NFL's the conduction between which is incoherent. Experimentally, the "coherence" is clearly confined to the planes - whether the electrons are or are not is a different question. The challenge is to understand how the incoherence of *c*-axis transport, or the confinement of coherence to the *ab* planes, comes about. In seeking to do so, we believe that there are two key facts to exploit, namely: (1) electronic correlations are strong, and (2) $t_\perp/t_\parallel$ is small, *i.e.* there is a significant anisotropy of nominal bandwidths. In particular, we will take $t_\perp \ll t_\parallel$ and $t_\parallel$ of order the interaction energy scale.

It is important to emphasize that anisotropy alone cannot account for the qualitatively anomalous c-axis conductivity in all of the HTSC's. Apart from theoretical arguments, there is a classic experimental "proof" of this point. Sr_2RuO_4, a structural analogue of La_2CuO_4, is predicted by band structure calculations to be about as anisotropic as, if not more than, La_2CuO_4 [28]. In Sr_2RuO_4, however, ρ_c and ρ_{ab} display the *same* temperature dependence at low temperatures, with an anisotropy of approximately 500 [29]. Moreover, the temperature dependence is of Fermi liquid form, $\rho = A + BT^2$. While correlation effects in Sr_2RuO_4 are not negligible [29], as evidenced by specific heat and magnetic susceptibility measurements, they are simply not as *qualitatively* severe as those in La_2CuO_4, for Sr_2RuO_4 is a good metal, while La_2CuO_4 is a Mott insulator! Thus, the experimental situation with respect to Sr_2RuO_4 strongly supports the idea that the anomalous c-axis conductivity in the cuprates results from a conspiracy of both sufficiently high anisotropy and sufficiently strong correlations. Paranthetically, we note that the superconducting transition temperature in Sr_2RuO_4 is only $1K$, which is consistent with the interlayer tunneling scenario for the HTSC's, a scenario which associates high T_c with the absence of coherent c-axis conduction

in the normal state.

We are thus led into the following conundrum: if the correct fixed point (ignoring the superconducting transition) governing the physics of the HTSC's is not an (anisotropic) 3D Fermi liquid, just what is the appropriate fixed point? In contemplating this question, we should recognize that the vast majority of theoretical approaches toward understanding the normal state have been based on strictly 2D models. In doing so, there is an implicit assumption that c-axis transport of electrons is somehow unimportant, but the question is *why and in what sense* is it "unimportant"?

One sense in which the hopping would be unimportant is realized in the opposite extreme from a 3D Fermi liquid: a genuine 2D NFL. Such a state would be possible if the interplane hopping operator was renormalization group (RG) irrelevant. While this is not the case if the coupled 2D liquids are Fermi liquids, one expects quite generally that the scaling dimension of $t_\perp$ and its renormalization group status can be altered by interaction effects in a NFL. However the observation that the HTSC's exhibit sharp Fermi surfaces is sufficient evidence to establish that, despite such renormalization, $t_\perp$ remains a relevant operator. For example, if the singularity of $n(k)$ near k_F may be parametrized by

$$\frac{\partial n(k)}{\partial k} \sim -|k - k_F|^{2\alpha - 1} \tag{5}$$

then at the tree level of the RG, $t_\perp$ is irrelevant if and only if $2\alpha > 1$. But in this case there would not be a sharp Fermi surface at all because $\partial n(k)/\partial k$ would not diverge as $k \to k_F$! A sharp Fermi surface therefore implies the relevance of $t_\perp$. A more general, parametrization independent result is that $t_\perp$ is relevant (at least at the tree level) if the integral over energy of the electron spectral function, $\rho(\mathbf{k}, \omega)$, for some momentum $\mathbf{k}$ at the Fermi surface, from zero energy out to a cutoff energy Λ, does not vanish faster than Λ. The presence of a "quasiparticle" peak in the angle resolved photoemission data is thus, in and of itself and independent of model or interpretation, enough to demonstrate the relevance of $t_\perp$ [30]. The "confinement" of the electrons to the planes in the usual renormalization group sense is therefore ruled out.

The key theme which will run through this paper, however, is that the *relevance* of $t_\perp$ in the RG sense, and the ensuing deconfinement of the electrons, *do not guarantee* that single particle interliquid hopping will be *coherent, i.e.* the deconfinement of coherence. Our proposal, then, is that there exists a new state which is in some sense an intermediary between strictly 2D and 3D coherent states. In this state there is a macroscopic number of 2D NFL's coupled by single particle hopping which is not RG irrelevant, but the interliquid hopping is rendered incoherent by interaction effects. We refer to this state as a state of "confined coherence", and schematically denote it by $(2\mathrm{D})_{\mathrm{NFL}} \otimes (1\mathrm{D})_{\mathrm{incoh}}$. The coherence which is clearly absent experimentally is the coherence in the finite temperature transport, but we believe this to be the result of the loss of quantum coherence even in the pure system at zero temperature. The defining feature of a state with confined coherence, then, is this incoherence at zero temperature and in the pure material. Our definition of the incoherence of the hopping is essentially the fundamental definition of quantum coherence: the hopping is incoherent if it is impossible, even in principle, to observe intereference effects between histories which involve the motion of particles between the incoherently coupled NFL's. The coherence persists in the individual NFL's because interference effects between different

histories in which no particle hops between different NFL's remain in principle observable. Therefore, the coherence can be said to be confined to the individual NFL's. This definition connects our proposal for an explanation of the anomalous c-axis transport in the cuprates directly to the decohering histories approach to the quantum to classical crossover and the two level system problem [31]. Also, although this work was originally motivated, as described above, by the peculiar c-axis conductivity in the cuprates, our conclusions will be relevant to *any* sufficiently anisotropic and sufficiently strongly correlated metal, and the above definition is the natural one for making contact with certain mysterious experimental results on the organic conductor $(TMTSF)_2PF_6$, which we now discuss.

We believe that, apart from our original motivation from the cuprate data, the need for a state with confined coherence is also clear in the experiments on the organic conductor $(TMTSF)_2PF_6$ of Kang, *et al* [32] and Danner, *et al.* [33]. As we have pointed out elsewhere [7], this material exhibits an exotic low temperature metallic phase in which magnetoresistance depends only on the component of the magnetic field perpendicular to its *ab* planes. The experiments on this material will be discussed in detail in Section VIII, where we argue that such an anisotropy can result only if hopping between different *ab* planes is irrelevant or if it is incoherent. The essential point is that if the hopping is incoherent, then interference effects between histories where particles hop between planes cannot be observed due to the randomization of the relative phase of the histories: in this case, the flux through these paths has no effect on physical observables (other than through Zeeman effects) and magnetoresistance should only be a function of the field normal to the *ab* planes. Note that this is experimentally the case even for resistivity *perpendicular* to the *ab* planes which is totally impossible in any semiclassical description. The only possible alternative explanation would be that the flux enclosed by paths which leave the *ab* plane is unimportant because the electrons are effectively confined and don't leave the *ab* planes. However, conduction perpendicular to the *ab* planes is not insulating at low temperatures and so the renormalization group irrelevance of the hopping appears to be excluded as an explanation.

Further, a direct test of Danner, *et al.* [33] of the coherence of *c*-axis transport demonstrated that, in addition to a phase where $(TMTSF)_2PF_6$ exhibits a three dimensional Fermi surface and coherent *c*-axis transport, the material has another metallic phase where no signs of coherent transport out of the *ab* planes are present and no evidence for a three dimensional Fermi surface exists. The two phases are separated by only an order one change in the magnitude of the conductivity out of the *ab* plane, which is hardly what one expects if the transverse hopping is relevant in one phase and irrelevant in the other. The incoherence effects were most pronounced at the lowest temperatures studied and in the cleanest samples. These experiments provide direct evidence that the confinement of coherence does occur and that it is a truly new state of matter, rather than a reiteration of known possibilities for the irrelevance of $t_\perp$ or an impurity or finite temperature effect.

C. Theoretical Motivation

Motivated by the apparent experimental need for a state with coherent transport confined to the *ab* planes, it is a natural question whether there is any theoretical foundation for believing in the existence of such a state. We believe that there is in fact a strong foundation,

which we will lay out in detail in Sections II-V; here we give a brief sketch of the underlying physics.

We begin with the same starting point as RG calculations for the relevance/irrelevance of interchain hopping: the recognition that, with a truly 1D system, in turning on weak interchain hopping one really has, in a sense, a strong coupling problem to begin with in that the zero hopping state is not a Fermi liquid. If the intrachain correlation strength is much larger than the interchain hopping, it is not *a priori* appropriate to consider the electron-electron interaction as a perturbation on an anisotropic free Fermi gas. Rather, one should consider the *interliquid hopping* as the perturbation, which opens up the possibility of finding a 2D NFL by perturbing (in the interchain hopping) about the 1D Luttinger liquid. Similarly, suppose one began with a 2D NFL metal, and then coupled a macroscopic number of these by interliquid hopping to form a 3D system. Although increasing the dimension of a system is expected to help stabilize a Fermi liquid, again, if the interliquid hopping is much smaller than the effective correlation strength in the 2D NFL starting state, one cannot rule out the possibility of an anisotropic NFL state resulting. It is exactly this path which has led to the proposal that the interliquid hopping could be renormalization group irrelevant, resulting in the stabilization of some non-Fermi liquid groundstate. It is generally accepted that this is the only non-trivial way to avoid a crossover to three dimensional Fermi liquid behavior in the low temperature limit, however, this belief does not result from any demonstration that this is the case, but rather from a simple lack of alternatives. We believe that the experiments we have discussed require another alternative *viz.*, that the hopping in one or more directions be intrinsically incoherent.

Recall that the perturbative irrelevance of $t_\perp$ results if the NFL states coupled by it have the property that the low energy spectral weight of the electron creation and annihilation operators vanishes rapidly enough. It is formally equivalent to the infrared convergence of perturbation theory in $t_\perp$ due to the small matrix elements of $t_\perp$ to states with small energy denominators. In order to give a hint of why states with intrinsically incoherent hopping might exist, consider the perturbation theory associated with the problem of making a 2D "stacking" of 1D electron liquids. The simplest form of interliquid hopping operator will remove an electron of momentum k from one liquid and insert it into an adjacent one. If the 1D liquids are free Fermi gasses, then an electron state of momentum k is a single particle, energy eigenstate and is *degenerate* with each of the electron states of the same momentum in all of the other liquids. It is therefore clear that perturbation theory in the interliquid hopping is degenerate perturbation theory and that, in this case, $t_\perp$ is relevant and, since the perturbation theory was degenerate, one should select states which diagonalize the interliquid hopping, and then perturb in the interactions. These states are built out of the creation and annihilation operators for Bloch states with well-defined interliquid momenta $k_\perp$, and such states will be energy eigenstates. As we will discuss, in the case of coupled Fermi liquids interactions do not change things in any essential way. Therefore, coherence is expected in both cases and the interactions are always to be treated as a perturbation.

On the other hand, if the 1D liquids are NFL's (which is actually always the case) then the interliquid hopping operator which creates a hole of momentum k in one liquid and a particle of momentum k in an adjacent liquid is no longer a zero energy operator. This is because an electron (or hole) of momentum k is *not* an energy eigenstate. Rather, it is a (precisely specified) superposition of energy eigenstates and does not, therefore, have a precise energy.

Imagine that the spectral function for the interliquid hopping were to be a flat function of energy: in this case the perturbation theory is still infrared divergent since the matrix elements between low energy states are vanishing too slowly. However, the perturbation theory is no longer degenerate since the states connected to the ground state have energies which are widely different and if anything, one's prejudice is that Fermi's Golden Rule might be a more appropriate starting point than Bloch states. The non-degeneracy of the hopping will impede, perhaps destroy, the formation of interliquid Bloch states. Roughly speaking, if there is some way to characterize the "width" in energy of the eigenstates which are in the superposition composing the in-liquid electron state of momentum k, then if this width is much smaller than the interliquid hopping rate it makes sense to perform degenerate perturbation theory, form interliquid electronic Bloch states first, and *then* consider the effect of those terms which led to the width as a perturbation on the (anisotropic) 2D free Fermi gas. On the other hand, if the width is much larger than the interliquid hopping rate, then perturbation theory in the hopping is non-degenerate and coherent interliquid electronic Bloch states are not a sensible starting point. Interliquid transport could actually be intrinsically incoherent, *i.e.* diffusive, due to the lack of a well-defined interliquid velocity at the Fermi surface. In fact, the Fermi surface will not show a 2D character at all, since that whole picture is based on the existence of the 2D Bloch states. *Intra*liquid transport may be metallic, but *inter*liquid transport will not be.

In essence, the in-liquid interactions responsible for destroying the Fermi liquid state give an electron of momentum k an intrinsic "lifetime" (although, as we shall see, the term 'lifetime', in its usual sense, is not precisely correct: the electron spectral function is power law, not Lorentzian) which, if shorter than the timescale for interliquid hopping, will render all interliquid transport completely incoherent. It is important to emphasize that this "lifetime" is not thermally, nor impurity, induced: it occurs in a *pure* system at *zero temperature*.

As we will discuss, for the case of coupled one dimensional chains, this physics is directly related to the physics of the two level system, a model of a single two state degree of freedom (a "spin") coupled to a bath of harmonic oscillators. The two level system model is the prototype for studies of quantum coherence effects, and there is believed to exist a regime at zero temperature in which flipping of the spin is purely incoherent due to the effects of the harmonic oscillator enviroment. The tunneling is incoherent in the sense that it is impossible even in principle to observe interference effects between histories of the entire system in which the history of the spin is different. The term in the Hamiltonian which flips the spin remains a relevant perturbation but the incoherent regime is clearly qualitatively different from the coherent regime, where interference effects are in principle observable. The existence of the incoherent regime was the main theoretical motivation for our proposal that a similar regime could exist for hopping between non-Fermi liquids and the details of the connection between the two problems form one of the central sections of this paper (Section II). For fermionic hopping between chains, the incoherence of the single particle hopping would clearly preclude Bloch states and the formation of a two dimensional Fermi surface; it can therefore be expected to result in different long time, low energy properties and constitutes a different fixed point from any at which two dimensional coherence obtains. There is, however, no real reason why an analogous proposal cannot exist for coupled two dimensional non-Fermi liquids, the problem that is ultimately connected to the experiments.

Unfortunately, our understanding of 2D NFL states is not well enough developed to permit precise calculations to be made. However, it is possible to consider the analogous problem in one less dimension, namely, the problem of coupled 1D electron liquids. It is well known that interacting electrons in 1D are always NFL's. A very general (non-insulating) class of these go by the name of "Luttinger liquids". We therefore consider the problem of Luttinger liquids coupled by interliquid single particle hopping as a potential paradigm for coupled NFL's. This problem is far from new: as early as 1974, in the context of the newly discovered quasi-1D organic conductors, Gorkov and Dzyaloshinskii discussed how various key properties of a 1D electron "chain" could be destroyed by the presence of interchain hopping [34]. More recently, many other authors have addressed the problem, using various techniques [35]. To our knowledge, however, ours is the only approach which directly addresses the question of interliquid *coherence*. This question is of crucial importance, for many of the other approaches *begin* with an anisotropic 2D electron gas (or its two chain analogue), a state with manifestly coherent interliquid hopping, upon which interactions are treated perturbatively. In those approaches which do not begin with the anisotropic 2D electron gas, we believe that, while in some cases unreasonable approximations and/or errors have been made, in general, these works have all correctly demonstrated the relevance of $t_\perp$ but have simply not addressed the question of its coherence. In general, past workers have argued that the flow away from $t_\perp = 0$ should lead to higher dimensional coherence and, for infinitely many chains, to a Fermi liquid or to some other (CDW, SDW or BCS) known higher dimensional fixed point, mainly because of the lack of an alternative proposal. We believe that incoherent hopping does constitute such an alternative, that some alternative is required by the experimental situation, and that the experiments, particularly in the organic material $(TMTSF)_2PF_6$, strongly support our proposal.

We emphasize again that we are interested in the *weak interliquid hopping regime* where the bare interliquid hopping parameter is much smaller than both the bare intraliquid hopping rate and the bare intraliquid interactions. This is not to say that we demand that perturbation theory in the interliquid hopping must be convergent, indeed quite the contrary, as shall be discussed further below.

As we have mentioned, the key construct needed in investigating the nature of interliquid hopping is the electron spectral function, $\rho(k,\omega)$, defined by

$$\begin{aligned}\rho(k,\omega) &= \theta(\omega)\rho^+(k,\omega) + \theta(-\omega)\rho^-(k,-\omega) \\ &= \sum_n \left\{\theta(\omega)|\langle n_{N+1}|c_k^\dagger|0_N\rangle|^2\delta(\omega - E_n^{N+1}) + \theta(-\omega)|\langle n_{N-1}|c_k|0_N\rangle|^2\delta(\omega - E_n^{N-1})\right\}\end{aligned}$$

where $|0_N\rangle$ is the ground state of the N-particle system, and $|n_{N\pm1}\rangle$ are energy eigenstates of the $N \pm 1$-particle system. In a FL, $\rho(k,\omega)$ is dominated by a term which sharpens up to a δ-function as $k \to k_F$. This term, of weight $Z_k \neq 0$, is the quasiparticle part of $\rho(k,\omega)$. The remainder of $\rho(k,\omega)$ is featureless so that, as far as low energy properties are concerned, only the quasiparticle part of $\rho(k,\omega)$ matters. The physical interpretation is that an electron inserted into (removed from) a FL with momentum k not too far above (below) k_F propagates coherently with a sharp energy, the quasiparticle energy. If $\langle i,j\rangle$ label physically adjacent liquids, and k in-liquid momenta, then an interliquid hopping term of the form

$$H_\perp = t_\perp \sum_{\langle i,j\rangle,k} \left(c^\dagger_{i,\sigma}(k) c_{j,\sigma}(k) + \text{h.c.} \right)$$

will directly couple a quasiparticle state in one liquid with an *energy degenerate* quasiparticle state in the physically adjacent liquids, leading to the formation of interliquid Bloch states of precise interliquid momenta. An interliquid band will therefore form, entailing a coherent interliquid velocity and hence coherent interliquid transport.

In contrast, in a Luttinger liquid (or any NFL, by definition) there are no Landau quasiparticles. The quasiparticle weight, Z_k, is zero, but in a nontrivial way. It is not simply a matter of taking the FL spectral function and sending Z_k to zero, for that would result in a completely featureless spectral function. A "metallic" electron liquid must have singularities in $\rho(k,\omega)$ in order to exhibit coherent in-liquid transport. The Luttinger liquid spectral function differs from that for a FL in that its singularities are power law in nature, even at the Fermi surface. For the physically most relevant case of spin-independent electronic interactions, $\rho(k,\omega)$ has singularities at $\omega = \pm v_c k, v_s k$ determined by a single exponent α, and v_c and v_s denote velocities of propagation of charge and spin currents. There is a singularity in $n(k)$ at $k = k_F$ which is precisely as given in (5). The nontrivial regime is therefore $\alpha < 1/2$. Although $t_\perp$ is formally RG relevant in this regime, we emphasize that there is no *a priori* reason for the boundary $\alpha = 1/2$ separating relevant from irrelevant $t_\perp$ to be the same boundary for separating coherent from incoherent interliquid hopping. Quite generally, the issue of whether an operator is relevant or irrelevant is simply a question of whether perturbation theory in that operator is divergent or convergent. The problem of coherence versus incoherence is a more subtle issue involving the *way* in which perturbation theory diverges (and therefore always involves relevant perturbations). As we have mentioned, a simple model within which to discuss quantum coherence and incoherence is that of a two level system (TLS) coupled to a dissipative bath, and this problem is deeply connected to our own. We will therefore begin by presenting a detailed discussion of the physics of the TLS in a way which will naturally generalize to the problem of determining the nature of interliquid hopping between Luttinger liquids.

The remainder of the paper is set out as follows. In Sec. II, we present a detailed and germane discussion of the physics of a two level system coupled to a dissipative bath, which provides an illustrative zero-dimensional analogue of the physics we wish to discuss. Sections III-V form the heart of the paper, in which a detailed qualitative and quantitative discussion of the problem of weakly coupled Luttinger liquids is presented. In sections VII and VIII, respectively, we present detailed arguments of why a state of confined coherence provides the only plausible way to account for several anomalous aspects of the physics of the HTSC's and of the (highly anisotropic) organic conductor $(TMTSF)_2PF_6$. The cuprate case is perhaps of more interest to the general reader, however the organic case furnishes quite compelling experimental support for our proposal. A final conclusion is given in Sec. IX.

II. INCOHERENCE AND THE TWO LEVEL SYSTEM

We will use the term "incoherent" in this paper in the same sense in which it was used in discussions of the two level system, or Caldeira-Leggett, problem [31]. In fact, much of what

we propose is most clearly explained by analogies to that model and for that reason we now give a self-contained, idiosyncratic review of the model and some of its known properties. We will use a somewhat unusual approach to the model which is considerably less elegant than some existing ones but which we believe to be transparent and, more importantly, generalizable to the problem of fermionic hopping.

To begin with we define, following Ref. [31], the two level system model with the Hamiltonian:

$$H_{\rm TLS} = \frac{1}{2}\Delta\sigma_x + \frac{1}{2}\epsilon\sigma_z + \sum_i \left(\frac{1}{2}m_i\omega_i x_i^2 + \frac{1}{2m_i}p_i^2\right) + \frac{1}{2}\sigma_z \sum_i C_i x_i \tag{6}$$

Here C_i is the coupling to the ith oscillator, and m_i, ω_i, x_i and p_i are the mass, frequency, position and momentum of the ith oscillator, respectively.

We restrict our discussion of the model to zero temperature and the so called ohmic regime [31] where the spectral density of the bath is given by:

$$\begin{aligned} J(\omega) &= \frac{\pi}{2}\sum_i \frac{C_i}{m_i\omega_i}\delta(\omega-\omega_i) \\ &= 2\pi\ \alpha\ \omega \exp(-\omega/\omega_c) \end{aligned} \tag{7}$$

α is a positive constant measuring the strength of the coupling to the bath and ω_c is a cutoff frequency. The α here should not be confused with the α which we have previously introduced in our discussion of the Fermi surface in the cuprates as defining the anomalous exponent of the single particle Green's function.

The two level system model describes a single quantum mechanical degree of freedom which can be in either of two states and which is coupled to a bath of harmonic oscillators. We are primarily interested in the $\epsilon = 0$ case and subsequent discussion refers to this case unless otherwise stated. In this case, one may think of the model as one in which a particle tunnels with tunneling matrix element $\Delta/2$ between two degenerate states (labelled by $\sigma_z = \pm 1$). The environment, represented by the bath of oscillators, influences the tunneling because the bath is sensitive to which of the states the spin is in. Hereafter we will refer to the discrete degree of freedom as a "spin" for convenience. This is appropriate since we are describing the spin with Pauli matrices.

The model provides the prototypical example of a quantum to classical crossover, since for $C_i = 0$ the model represents the quantum mechanics of an isolated two state system, whereas for sufficiently strong coupling to the environment the dynamics of the spin, if followed without reference to the oscillator bath, are dissipative and no quantum coherence effects are observable [31]. In fact, this is how one generally expects classical behavior to emerge for macroscopic systems: the macroscopic degrees of freedom exchange energy with an enormous number of unobserved microscopic degrees of freedom and therefore different histories are unable to maintain a definite relative phase long enough for quantum interference effects to manifest themselves. This is one possible solution to the famous Schrödinger's cat problem of quantum mechanics, in which apparent paradoxes arise as a result of the fact that while macroscopic objects are never observed to be in superpositions of different states, such superpositions are, in principle, possible according to the standard Copenhagen interpretation of quantum mechanics. The resolution to this problem relies on the fact that,

since it is impossible to measure an object in a superposition of states with respect to the measured quantity, the real meaning of observing an object in a superposition of states is making an observation which is only compatible with the object having been in a superposition of states in the past, *i.e.* an interference experiment. For example, in the Young double slit experiment the particle is not observed to be in a superposition of having passed through each slit and, in fact, if any measurement is made of it passing through either slit then it is not in any such superposition. However the interference pattern which results at the screen behind the slits is only possible because the particle was in a superposition of having passed though both slits before being measured at the screen. Since the only meaningful way to detect superpositions is through such interference experiments, a definite phase relationship between the different histories contributing to the final state is required. Hence, the interchange of energy with the environment can totally remove the possibility for such observations and is expected to do so for macroscopic objects or variables (except under very special circumstances, for example in certain Josephson junction experiments [31]).

What sort of quantum interference effects do we expect to be able to observe in the TLS for sufficiently weak coupling to the environment? Consider a model where the coupling to the environment vanishes, i.e. $\alpha = 0$, and the system is prepared in a state where the spin is in a σ_z eigenstate. The exact eigenstates of the spin are the σ_x eigenstates which are split by an energy Δ so that the initial state of the system is a superposition of these two states of different energy with a definite phase between the two states in the superposition. Since the two states have different energies, this phase is not time independent (the phases evolve differently for the history where the particle is in the symmetric state and the history where it is in the antisymmetric state). Furthermore, for vanishing coupling to the environment the relative phase remains well defined indefinitely. The time dependence of the phase therefore results in observable oscillations in the expectation value of σ_z, in fact (in units where $\hbar = 1$) $\langle \sigma_z(t) \rangle = \cos \Delta t$. The oscillations are a quantum interference effect between histories which involve the spin spending different amounts of time in the various σ_z states. In general, we would expect such oscillations to also occur when the spin is coupled to the environment provided the spin is capable of flipping without exchanging an amount of energy with its environment sufficient for the randomization of the phase of the history involving the spin flip. As we shall see, in the TLS model the oscillations persist for a range of couplings to the environment, albeit with coupling dependent damping of the oscillations. Such damping results from the exchange of energy between the spin and the environment during the course of a typical flip.

To study these oscillations the standard theory of the TLS focuses on the quantity $P(t)$, which is the probability of finding the system in the $\sigma_z = 1$ state for $t > 0$ for a system which has been prepared by clamping the spin into the $\sigma_z = 1$ state for all $t < 0$, allowing the oscillator bath to relax to equilibrium in this configuration, and finally releasing the spin at $t = 0$. Note that the calculation of $P(t)$ is equivalent to determining $\langle \sigma_z(t) \rangle$, since the two are simply related by $\langle \sigma_z(t) \rangle = 2P(t) - 1$ [36]. $P(t)$ is an appropriate quantity to study for questions about macroscopic quantum coherence since, if the spin represents a generic macroscopic quantum degree of freedom which the experimenter can observe and control, whereas the oscillators represent microscopic degrees of freedom which are beyond both control and observational capacities of the experimenter, it is exactly the sort of preparation

used in the definition of $P(t)$ which is possible experimentally. The signature of quantum coherence in $P(t)$ will be the presence of oscillations (damped or otherwise) in contrast to the incoherent relaxation, $P(t) \sim \frac{1}{2}(1 + e^{-\Gamma t})$, which should result if the spin is sufficiently strongly coupled to the environment that the relative phases associated with histories in which tunneling between the two σ_z eigenstates occurs are randomized. A transition to purely incoherent relaxation is indeed found to occur in the TLS problem, even at short times, when $\alpha > 1/2$ [38].

For further discussion of the TLS problem, it is convenient to make a canonical transformation on the original model by taking

$$H'_{\rm TLS} = \hat{U} H_{\rm TLS} \hat{U}^{-1} \tag{8}$$

where

$$\hat{U} = \exp\left(-\frac{1}{2}\sigma_z \sum_i \frac{C_i}{m_i \omega_i^2} \hat{p}_i\right) \tag{9}$$

$\hat{p}_i$ is the momentum operator of the ith oscillator. The new Hamiltonian takes the form:

$$H'_{\rm TLS} = \frac{1}{2}\Delta(\sigma^+ e^{-i\Omega} + h.c.) + H_{\rm oscillators} \tag{10}$$

where $\Omega = \sum_i \frac{C_i}{m_i \omega_i^2} p_i$. The coupling to the oscillators has been removed by the transformation, but in the process the tunneling operator between the two states has been replaced by an operator which creates and destroys excitations of the oscillator bath (p_i is expressible in terms of the creation and annihilation operators for the excitations of the ith oscillator) as well as changing the state of the spin. It is clear that quantum oscillations of $P(t)$ will be in danger of being destroyed should the low-frequency oscillator density of states and/or the low-frequency couplings C_i be sufficiently large. Again we consider only the ohmic case defined by Eq. 7 so that the coupling is parametrized by a single dimensionless number, α; the oscillations will be called into doubt for large α.

In this formulation, $P(t)$ can be reinterpreted as the probability of finding $\sigma_z(t) = 1$ for a system in which Δ is suddenly switched on at time $t = 0$ with the system in the $\Delta = 0$ groundstate with $\sigma_z = 1$. The previous definition in which the spin was clamped in the $\sigma_z = 1$ eigenstate for all negative times and the oscillators were allowed to adapt to the clamped state is equivalent.

We begin our discussion of the physics of this model with the two point correlation function of $\sigma^+ e^{-i\Omega}$, which obeys

$$\begin{aligned} \langle \sigma^+ e^{-i\Omega(t)} \sigma^- e^{i\Omega(0)} \rangle &= \exp\left\{-\int_0^\infty \frac{1 - e^{-i\omega t}}{\omega^2} J(\omega)\right\} \\ &= \exp\left\{-2\alpha \int_0^\infty \frac{1 - e^{-i\omega t}}{\omega} e^{-\omega/\omega_c}\right\} \\ &\sim e^{i\pi\alpha} (\omega_c t)^{-2\alpha} \end{aligned} \tag{11}$$

From the correlation function we can immediately construct the spectral function of the operator $e^{i\Omega}$ in the low energy, universal regime:

$$\rho_\Omega(\omega) = \sum_m |\langle m|e^{i\Omega}|GS\rangle|^2 \delta(\omega - E_m) \tag{12}$$
$$= \Gamma^{-1}(2\alpha)\ \theta_+(\omega)\ \omega^{-1+2\alpha}\omega_c^{-2\alpha}\exp(-\omega/\omega_c)$$

where $\{m\}$ is a complete set of oscillator eigenstates with energies E_m and $|GS\rangle$ is the oscillator ground state. The spectral function is normalized to integrate to unity since $\langle e^{-i\Omega(t)}e^{i\Omega(t)}\rangle = 1$.

The short time approximation to $P(t)$ can be constructed straightforwardly using the spectral function above and ordinary time dependent perturbation theory. We find:

$$P(t) = 1 - \frac{\Delta^2}{2}\int d\omega \rho_\Omega(2\omega)\frac{\sin^2(\omega t)}{\omega^2} + \cdots \tag{13}$$

Notice that when $\alpha > 1$, $\rho_\Omega(\omega) \sim \omega^{-1+2\alpha}$ results in an infrared convergent $P(t)$; in the limit $\Delta \to 0$, $P(t) \to 1$ for all t. This corresponds to the irrelevance of Δ and the localization of the spin predicted by Chakravarty and Bray and Moore [37] based on a mapping of the TLS to the inverse squared Ising model.

Conversely, for $\alpha \to 0$, $\rho_\Omega(\omega) \to \delta(\omega)$ and $P(t) = 1 - \frac{\Delta^2}{4}t^2 + ...$, in agreement with the expansion of the exact result $P(t) = (1 + \cos\Delta t)/2$. For $0 < \alpha < 1$ we are in a more complicated region. Clearly the difference between $P(t)$ and 1 grows to order unity for any arbitrarily small Δ throughout this region (this simply reflects the renormalization group relevance of Δ) and one would at first sight be tempted to conclude that throughout this region $P(t)$ would undergo damped oscillations with a period approximately given by $t_{\rm osc}$, where $t_{\rm osc}$ satisfies:

$$1 = \frac{\Delta^2}{2}\int d\omega \rho_\Omega(2\omega)\frac{\sin^2(\omega t_{\rm osc})}{\omega^2} \tag{14}$$

One should be cautious, however, in view of the fact that for $\alpha > 1/2$ the spectral function for the tunneling operator is vanishing at low frequencies and, at $\alpha = 1/2$, it is flat and featureless out to the cutoff scale. A flat spectral function is equivalent to a featureless density of states and is exactly the condition under which the approximation of Fermi's Golden Rule should be valid, implying incoherent decay without any recurrence effects or oscillations. We may scale out the time dependence in (13) to obtain to $O(\Delta^2)$ and for $\omega_c t \gg 1$

$$P(t) \approx 1 - 2^{2\alpha-1}\alpha\Delta^2\ \omega_c^{-2\alpha}t^{2-2\alpha}\int_0^\infty dx \frac{\sin^2 x}{x^{3-2\alpha}} \tag{15}$$

(for simplicity, we have replaced the cutoff $e^{-\omega/\omega_c}$ by a hard cutoff at ω_c). Thus, for $\alpha > 1/2$, where the spectral function for the tunneling operator is vanishing at low frequencies, we see that the $O(\Delta^2)$ term in $P(t)$ grows even *more slowly* than t, suggesting an even "more incoherent" decay of $P(t)$. If we define $\Gamma(t) = -dP(t)/dt$, the rate at which the spin flips, then in this regime $\Gamma(t)$ is bounded for all t. For the special value $\alpha = 1/2$, $\Gamma(t) = \Gamma$, a constant, and a naive re-exponentiation of the Golden Rule is $P(t) = (1 + e^{-\Gamma t})/2$, corresponding to $\langle\sigma_z(t)\rangle = e^{-\Gamma t}$. For $\alpha > 1/2$ it would appear reasonable to expect exponential relaxation, too. A self-consistent approximation to determining the relaxation

rate Γ for small Δ/ω_c involves cutting off the ω-integral at $\omega \sim \Gamma$ to give $\Gamma \sim \Delta^2 \omega_c^{-2\alpha} \Gamma^{2\alpha-1}$ yielding

$$\Gamma \approx \Delta \left(\frac{\Delta}{\omega_c}\right)^{\alpha/(1-\alpha)}$$

The right hand side is in fact nothing but $\Delta_{\rm ren}$, the renormalized tunneling rate which emerges from an RG analysis.

The true behavior of $P(t)$ in the region $1/2 < \alpha < 1$ is actually not rigorously known [31], but there are reasons for believing that the self-consistent argument given above is not too far from the truth. The true decay of $P(t)$ is probably not simply exponential relaxation, but the key point is that there are not any oscillations. Thus, despite the fact that the naive RG approach yields the same scale $\Delta_{\rm ren}$ as the self-consistent approach, it fails to distinguish between an essentially *coherent* $\Delta_{\rm ren}$ and a completely *incoherent* one.

For $0 < \alpha < 1/2$, $\Gamma(t)$ is unbounded and any attempt at characterizing $P(t)$ by exponential relaxation fails, as indeed it must as $\alpha \to 0$. But what is the correct interpretation of the behavior $1 - P(t) \sim t^{2-2\alpha}$? And can we go beyond lowest order in Δ? The simplest calculation addressing these questions, a kind of random-phase approximation, is the so-called non-interacting blip approximation (NIBA). The reader is referred to [31] for the more standard approach to this approximation, which also reveals the reason behind its curious name. Here, we shall present a very simple derivation which has the advantage of making the key approximations of the NIBA explicit [39].

Consider the canonically transformed TLS Hamiltonian (10). The equations of motion of the spin are

$$\begin{aligned}
\partial_t \sigma_z &= -i\Delta\{e^{-i\Omega}\sigma^+ - e^{i\Omega}\sigma^-\} \\
\partial_t \sigma^\pm &= \mp \frac{i\Delta}{2} \sigma_z e^{\pm i\Omega}
\end{aligned}$$

The equations for $\sigma^\pm$ can be formally integrated

$$\sigma^\pm(t) = \sigma^\pm(0) \mp \frac{i\Delta}{2} \int_0^t dt' \sigma_z(t') e^{\pm i\Omega(t')} \tag{16}$$

which can then be substituted into the equation for $\partial_t \sigma_z$. Taking expectation values yields

$$\left\langle \frac{d\sigma_z}{dt} \right\rangle = -\Delta^2 \int_0^t dt' \, \langle \sigma_z(t') D(t-t') \rangle \tag{17}$$

where

$$D(t-t') = \frac{1}{2}\left\{ e^{-i\Omega(t)} e^{i\Omega(t')} + e^{i\Omega(t)} e^{-i\Omega(t')} \right\}$$

The NIBA amounts to *neglecting correlations* in the expectation value under the integral, *i.e.* the replacement

$$\langle \sigma_z(t') D(t-t') \rangle \to \langle \sigma_z(t') \rangle \langle D(t-t') \rangle$$

is made. Defining $D_0(t-t') \equiv \langle D(t-t')\rangle$, we arrive at the integral equation

$$\frac{d\langle\sigma_z\rangle}{dt} = -\Delta^2 \int_0^t dt' D_0(t-t')\langle\sigma_z(t')\rangle \tag{18}$$

The formal solution of (18) is

$$\langle\sigma_z(t)\rangle = \langle\sigma_z(0)\rangle \sum_{n=0}^{\infty}(-\Delta^2)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2n-1}} dt_{2n} D_0(t_1-t_2)D_0(t_3-t_4)\ldots D_0(t_{2n-1}-t_{2n}) \tag{19}$$

however the integral equation may also be solved by introducing Laplace transforms, $\mathcal{L}(f(t)) \equiv \hat{f}(s)$. Taking the spin to be up at $t=0$ gives

$$\hat{\sigma_z}(s) = \frac{1}{s+\Delta^2\hat{D}_0(s)} \tag{20}$$

Equation (20) is the key result of the NIBA. The determination of $\sigma_z(t)$ is reduced to the calculation of $D_0(s)$, which is essentially the Laplace transform of the propagator in equation (11), followed by the inverse Laplace transform of $\hat{\sigma_z}(s)$. From (11) it can be shown that

$$\hat{\sigma_z}(s) = \frac{s^{1-2\alpha}}{\left(s^{2-2\alpha}+\Delta_{\rm eff}^{2-2\alpha}\right)} \tag{21}$$

where

$$\Delta_{\rm eff} \equiv [\Gamma(1-2\alpha)\cos\pi\alpha]^{1/(2-2\alpha)}\Delta\left(\frac{\Delta}{\omega_c}\right)^{\frac{\alpha}{1-\alpha}}$$

is, up to a numerical prefactor, the renormalized hopping rate $\Delta_{\rm ren}$ as given by the RG calculation.

We now briefly discuss the various cases (more details are provided in [31]), beginning with the two simplest.

(i) $\alpha = 0$ - here we have $\hat{D}_0(s) = 1/s$, yielding the correct result for a free spin, $\langle\sigma_z(t)\rangle = \cos\Delta t$.

(ii) For $\alpha = 1/2$, $\hat{\sigma_z}(s) = (s+\Delta_{\rm eff})^{-1}$, whence $\langle\sigma_z(t)\rangle = \exp(-\Delta_{\rm eff}\,t)$. Of course, this corresponds precisely to the situation in elementary quantum mechanics where one would invoke Fermi's Golden Rule, there being a constant density of states for the system to "decay" into. Indeed, the NIBA is equivalent to simply exponentiating the decay rate $\Delta_{\rm eff}$. Again, we emphasize that at this value of α, Δ is RG relevant, yet clearly its effect is purely incoherent.

(iii) In the region $0 < \alpha < 1/2$ [38], $\hat{\sigma_z}(s)$ acquires a pair of poles on the principal Riemann sheet. A straightforward Laplace inversion yields the following contribution to $\langle\sigma_z(t)\rangle$

$$\langle\sigma_z(t)\rangle_{\rm poles} = \frac{1}{(1-\alpha)}\cos\left\{\cos\left[\frac{\pi}{2}\frac{\alpha}{(1-\alpha)}\right]\Delta_{\rm eff}\,t\right\}\exp\left\{-\sin\left[\frac{\pi}{2}\frac{\alpha}{(1-\alpha)}\right]\Delta_{\rm eff}\,t\right\} \tag{22}$$

There is also a branch cut in $\hat{\sigma_z}(s)$ which gives a nonoscillatory contribution to $\langle\sigma_z(t)\rangle$. Thus, $P(t)$ in this region of α is the sum of a damped oscillation, with oscillation frequency $\omega_{\rm osc} = \cos(\pi\alpha/(2-2\alpha))\Delta_{\rm eff}$ and damping $\Gamma = \sin(\pi\alpha/(2-2\alpha))\Delta_{\rm eff}$, and an incoherent background. As $\alpha \to 0$, one approaches the free spin limit, $\langle\sigma_z(t)\rangle = \cos\Delta t$, while as $\alpha \to 1/2$, the oscillation frequency $\omega_{\rm osc} \sim \pi(1-2\alpha)\Delta_{\rm eff} \to 0$, and the limit of purely exponential relaxation is reached.

(iv) For $1/2 < \alpha < 1$, the NIBA is not believed to be valid [31] because it predicts a non-universal power law decay of the σ_z two point function, in contradiction to the connection between the TLS problem and the Kondo problem. We give the NIBA result for completeness, however we believe that the true result in this regime, at least at intermediate times, is more likely exponential decay of $\langle\sigma_z(t)\rangle$, as at $\alpha = 1/2$. In any case, within the NIBA one finds that $\hat{\sigma_z}(s)$ has no poles on the principal Riemann sheet, and as a result $P(t)$ is purely incoherent [31],

$$\langle\sigma_z(t)\rangle = -\frac{\sin 2\pi\alpha}{\pi}\int_0^\infty dx\,\frac{x^{2\alpha-1}\,e^{-(\Delta_{\rm eff}\,t)\,x}}{x^2+2x^{2\alpha}\cos 2\pi\alpha + x^{4\alpha-2}} \tag{23}$$

The results of the NIBA are therefore consistent with the qualitative behavior one could intuit from the perturbative calculation of $1-P(t)$ to $O(\Delta^2)$. The key result is the existence of the phase $1/2 < \alpha < 1$ where Δ is a relevant, but incoherent, operator.

At this point, it is useful to make some observations regarding the TLS which help to emphasize the general physics of incoherence. In the TLS model, the important physical effect of finite α is that there is a substantial contribution to $P(t)$ from transitions to states with energies that are larger than the putative renormalized oscillation frequency, $\Delta_R \sim t_{\rm osc}^{-1}$ (see Eq. 14) . When the amount of weight in these transitions is larger than the amount of weight in transitions to low energy states, it no longer makes sense to consider the effects of Δ to be coherent. Effectively, each change of state, *i.e.* flipping of the spin, is accompanied by the creation or annihilation of a sufficient number of bosons in the environmental bath that the phase of that history is randomized compared to histories with no spin flip. Intuitively, *one has crossed over from degenerate or nearly degenerate perturbation theory to non-degenerate perturbation theory* (as opposed to the transition to irrelevant Δ where the long time perturbation theory becomes convergent). To illustrate the point, first note that we can calculate $\delta P(t) \equiv 1 - P(t)$ to $O(\Delta^2)$ exactly:

$$\begin{aligned}\delta P(t) &= \frac{\Delta^2}{2}\int_0^\infty d\omega\,\rho_\Omega(2\omega)\frac{\sin^2(\omega t)}{\omega^2} + \cdots \\ &= \Delta^2\,\Gamma^{-1}(2\alpha)\,t^{2-2\alpha}2^{2\alpha-2}\int_0^\infty d\theta\,e^{-2\theta/t}\theta^{2\alpha-3}\sin^2\theta + O(\Delta^4) \\ &= \Delta^2\frac{[1-(1+t^2)^{1-\alpha}\,\cos[2(\alpha-1)\arctan(t)]]}{4(1-2\alpha)(1-\alpha)} + O(\Delta^4)\end{aligned} \tag{24}$$

For times much longer than the inverse cutoff, $\omega_c^{-1} \equiv 1$, and for $0 < \alpha < 1$ we can use

$$\delta P(t) \sim \frac{\Delta^2}{4}\frac{\cos\pi\alpha}{(1-2\alpha)(1-\alpha)}\,t^{2-2\alpha} \tag{25}$$

If we now separate the contributions from the frequencies which are less than $t^{-1} = \Delta_R = \Delta^{1/(1-2\alpha)}$ from those that are greater than t^{-1} [40] then (again, for $t \gg \omega_c^{-1} \equiv 1$ and $0 < \alpha < 1$) we find that the low energy part contributes to $P(t)$ an amount

$$\begin{aligned}\delta P_{\rm low}(t) &= \frac{\Delta^2}{2}\int_0^{1/t} d\omega \rho_\Omega(2\omega)\frac{\sin^2(\omega t)}{\omega^2} + \cdots \\ &\sim \frac{\Delta^2}{2^{2-2\alpha}} t^{2-2\alpha}\Gamma^{-1}(2\alpha)\ \frac{{}_1F_2(-1+\alpha;\frac{1}{2},\alpha;-1)-1}{4(1-\alpha)} \end{aligned} \tag{26}$$

where ${}_1F_2$ is a generalized hypergeometric function [41]. This evaluates to $\frac{1}{4}\Delta^2 t^2$ at $\alpha = 0$ and, for $\alpha = 1/2$, to $\frac{1}{4}\Delta^2 t\,(2\,\mathrm{Si}(2) + \cos 2 - 1) \approx 0.45\Delta^2 t$ (Si is the sine integral function).

The high energy part contributes

$$\begin{aligned}\delta P_{\rm high}(t) &= \frac{\Delta^2}{2}\int_{1/t}^{\infty} d\omega \rho_\Omega(2\omega)\frac{\sin^2(\omega t)}{\omega^2} + \cdots \\ &\sim \frac{\Delta^2}{2^{2-2\alpha}} t^{2-2\alpha}\Gamma^{-1}(2\alpha)\left(\frac{1}{4(1-\alpha)} + \frac{\Gamma(2\alpha)\cos\pi\alpha}{2^{2\alpha}(1-2\alpha)(1-\alpha)} - \frac{{}_1F_2(-1+\alpha;\frac{1}{2},\alpha;-1)}{4(1-\alpha)}\right) \end{aligned} \tag{27}$$

For $\alpha \to 0$ the high energy part vanishes like $\Gamma^{-1}(2\alpha)$ (the prefactor is $\frac{1}{4}\Delta^2 t^2\left(\frac{3}{2} - \gamma - \ln 2 + \frac{1}{6}\ {}_2F_3(1,1;2,\frac{5}{2},3;-1)\right) \approx 0.1\Delta^2 t^2$), while for $\alpha = 1/2$ the result is $\frac{1}{2\pi}\Delta^2 t\,(1 + \pi - {}_1\,F_2(-\frac{1}{2};\frac{1}{2},\frac{1}{2};-1) \approx 0.34\Delta^2 t$, comparable to the low energy contribution.

Clearly, the high energy contribution is insignificant as $\alpha \to 0$ because of the divergence of $\Gamma(2\alpha)$ (the gamma function has a simple pole at 0) and, for any arbitrary division into "high" and "low", could always be made so by taking a suitably small α. One therefore expects to find coherence in the limit $\alpha \to 0$. On the other hand, for finite values of α the high energy part can be as important as the low energy part, depending upon our division into high and low energy integrals. Using the qualitatively reasonable division above, we see that for $\alpha = 1/2$ the two contributions are in fact comparable. In fact, for any division scheme involving energy scales small compared to the oscillator cutoff, the high energy part must dominate for some $\alpha < 1$, since, in the limit where $\alpha \to 1$, the high energy part diverges logarithmically like $\frac{\Delta^2}{2\omega_c^2}\ln(\omega_c t)$ while the low energy part is finite and given by $\frac{\Delta^2}{2\omega_c^2}(\gamma + \ln(2) - \mathrm{Co}(2)) \approx 0.85\ \frac{\Delta^2}{2\omega_c^2}$, where Co is the cosine integral function and γ is Euler's constant. The high energy part can therefore be made arbitrarily large compared to the low energy part for any arbitrary partition into high and low energy pieces as we approach $\alpha = 1$. The dominance of the high energy part does not necessarily imply that the quantum oscillations must cease entirely; it could be that the oscillations would persist but become arbitrarily heavily damped. However, when the high energy part has become of order one, the argument that oscillations should occur with a frequency $\omega_{\rm osc} \sim \Delta_R$ becomes unreliable and, in fact, as we have seen above, the conclusion of the NIBA (which is known to be correct at $\alpha = 1/2$ from the exact solution [31]) is that the oscillations vanish for $\alpha = 1/2$ [38].

The reason for the success of our perturbation theory, which is essentially a "short time expansion", in predicting a qualitative change in the tunneling is that the expansion is valid out to precisely the time when the spin has order one probability of flipping and is therefore perfectly adequate to describe the nature of the states reached by spin flip processes. In

particular, it can identify whether these states are nearly degenerate with the initial state (and each other) or of widely disparate energies, which is the essential physical question for coherence. Hence, the main conclusions of this section: the qualitative behavior of $P(t)$, in the sense of whether or not it exhibits oscillations, *i.e.* quantum coherence, can actually be determined from lowest order perturbation theory. The special point $\alpha = 1/2$, at which the Golden Rule is naively applicable, separates a region of completely incoherent behavior, $1/2 \leq \alpha < 1$, from one of damped oscillations, $0 < \alpha < 1/2$.

III. THE CONNECTION TO FERMIONIC HOPPING

The existence of a third regime in the TLS problem with behavior qualitatively different from that occurring for irrelevant tunneling or undamped tunneling is suggestive, however, before we can claim that the lessons of the TLS have any relevance to the experimental peculiarities observed in, for example, the cuprates, we must make some firmer connection between the tunneling matrix element, Δ, and the interplane, single particle hopping, $t_\perp$. The experiments we are interested in involve coupling planes of interacting electrons together with a single electron hopping operator $O_{hop} = \sum_{\vec{k},\sigma,\langle ij\rangle} t_\perp(\vec{k}) c^\dagger_{\sigma,i}(\vec{k}) c_{\sigma,j}(\vec{k})$, where the sum over $\langle ij\rangle$ is a sum over nearest neighbor planes. Unfortunately, as we will see, any interesting outcome (an outcome other than three dimensional Fermi liquid theory) will require an interesting (i.e. non-Fermi liquid) starting point and there at present exists no solid non-Fermi liquid framework in two dimensions. We therefore study the problem of coupled one dimensional chains rather than planes. Once we have elucidated a connection to the TLS in this context and understand what is required to produce interesting physics in this case, we will be in a position to discuss what might be required for similar possibilities to be realized for coupled planes.

It is a not entirely trivial matter to determine *how* one should go about investigating the coherence/incoherence issue for coupled electron liquids. The essential question is, roughly speaking, whether or not we need to use degenerate perturbation theory in dealing with the action of the interliquid hopping term. Given that the excitation spectrum of a Luttinger liquid is gapless, one might at first think that degenerate perturbation theory is unavoidable. However, a moments reflection on the physics of the TLS will convince the reader that such is not necessarily the case.

We begin with the coupling of a pair of one dimensional, interacting electronic systems with a transverse hopping. We will study this problem using bosonization techniques [42,1]. General, gapless, one dimensional interacting electronic systems and higher dimensional Fermi liquids can both be studied via this approach so our results can be made at least that general.

The Hamiltonians of the isolated systems are of the general Luttinger liquid form:

$$\begin{aligned} H &= \frac{1}{4\pi}\int dx \ \left(v_\rho K_\rho (\partial\Theta_\rho)^2 + v_\rho K_\rho^{-1} (\partial\Phi_\rho)^2 + v_\sigma K_\sigma (\partial\Theta_\sigma)^2 + v_\sigma K_\sigma (\partial\Phi_\sigma)^2 \right) \\ &= \frac{1}{4\pi}\int dx \ \left(v_{\rho,N} (\partial\Theta_\rho)^2 + v_{\rho,J} (\partial\Phi_\rho)^2 + v_{\sigma,N} (\partial\Theta_\sigma)^2 + v_{\sigma,J} (\partial\Phi_\sigma)^2 \right) \end{aligned} \tag{28}$$

where K_ρ is interaction dependent and less than one for a repulsive interaction, while K_σ

is set to one hereafter as a consequence of considering only interactions which preserve the $SU(2)$ spin invariance. The bosonized form for the electron operator is given by

$$\Psi^{\dagger}_{\uparrow}(x) \sim \sqrt{\frac{\partial \Phi_{\uparrow}(x)}{\pi}} \sum_{m \ odd} A_m \ \exp\left(i[m\Phi_{\uparrow}(x) + \Theta_{\uparrow}(x)]\right) \tag{29}$$

so that the inter-liquid hopping is given by:

$$\begin{aligned}\Psi^{\dagger,(1)}_{\uparrow}(x)\Psi^{(2)}_{\uparrow}(x) \sim & \sqrt{\frac{\partial \Phi^{(1)}_{\uparrow}(x)}{\pi}} \sum_{m \ odd} A_m \ \exp\left(i[m\Phi^{(1)}_{\uparrow}(x) + \Theta^{(1)}_{\uparrow}(x)]\right) \\ & \times \sqrt{\frac{\partial \Phi^{(2)}_{\uparrow}(x)}{\pi}} \sum_{m \ odd} A^{\star}_m \ \exp\left(i[m\Phi^{(2)}_{\uparrow}(x) - \Theta^{(2)}_{\uparrow}(x)]\right)\end{aligned} \tag{30}$$

where

$$\Theta_{\uparrow} = 2^{-\frac{1}{2}}(\Theta_{\rho} + \Theta_{\sigma}) \tag{31}$$

$$\Theta_{\downarrow} = 2^{-\frac{1}{2}}(\Theta_{\rho} - \Theta_{\sigma}) \tag{32}$$

and

$$\Theta_{\rho}(x) = \Theta^0_{\rho} + N_{\rho}x/L - i\sum_{q\neq 0} |\frac{2\pi}{qL}|^{\frac{1}{2}} K_{\rho}^{-\frac{1}{2}} \ \mathrm{sgn}(q) e^{iqx} \left(b^{\dagger}_{\rho}(q) + b_{\rho}(-q)\right) \tag{33}$$

$$\Phi_{\rho}(x) = \Phi^0_{\rho} + J_{\rho}x/L - i\sum_{q\neq 0} |\frac{2\pi}{qL}|^{\frac{1}{2}} K_{\rho}^{\frac{1}{2}} \ \mathrm{sgn}(q) e^{iqx} \left(b^{\dagger}_{\rho}(q) - b_{\rho}(-q)\right) \tag{34}$$

where the b_{ρ} operators create and annihilate the bosonic, charge density eigenexcitations. Similar expressions obviously apply for Θ_{σ} and Φ_{σ}. The expression for hopping of down spin electrons is easily obtained by changing the sign of Θ_{σ} and Φ_{σ} in Eq. 30, while that for hops in the other direction can be obtained by interchanging the chain labels in 30.

In the above expressions the operators $\Theta^0_{\uparrow}$ and $\Phi^0_{\uparrow}$ are canonically conjugate to the the conserved quantum numbers $J_{\uparrow}$ and $N_{\uparrow}$ and are not expressible in terms of the bosons. The role of these operators was stressed by Haldane in his solution of the Luttinger model [1], however they are generally ignored since they do not enter into single particle correlation functions. They will be crucial for our discussion since it is the quantum numbers N and J that are analagous to σ_z in the TLS problem. This is readily apparent when the canonically transformed form for the tunneling matrix element, $\frac{\Delta}{2}\sigma^{+}e^{-i\Omega} + \ h.c.$ (see Eq. 10), is compared to the bosonized form for the interchain hopping in Eq. 30. Both contain operators which act to raise and lower otherwise conserved quantum numbers (σ_z for the TLS and $N_{\uparrow,1} - N_{\uparrow,2}$, $J_{\uparrow,1} - J_{\uparrow,2}$, etc. for the fermion hopping). In addition to the raising and lowering operators, both contain exponentials in bosonic creation and annihilation operators which are responsible for the interesting dynamics and determine the correlation functions of the operators. In fact, if the fermionic hopping occurred at only a single point in space, that

problem could be mapped onto the TLS problem. We do not believe that such a mapping exists for the physical model of a uniform interchain hopping. However, the formal connection between the models is quite strong and suggests that the interesting incoherent regime for the TLS might well have a fermionic analogue.

Let us now construct the analogue to $P(t)$ for the fermionic problem. As we argued above the natural analogy maps the Tomonaga bosons to the harmonic oscillator bath and the conserved quantum numbers N and J to σ_z. In this case a clear analogy exists to $P(t)$,as defined for the canonically transformed TLS Hamiltonian of Eq. 10. Instead of taking a system adapted to $\Delta = 0$ with $\sigma_z = 1$ and then turning on Δ suddenly, we adapt a system with some non-zero values for $N_{\uparrow,1} - N_{\uparrow,2}$, $J_{\uparrow,1} - J_{\uparrow,2}$, etc. to $t_\perp = 0$ and then turn on $t_\perp$ suddenly. Instead of studying the resulting oscillations (or lack thereof) in $\sigma_z(t)$ we study them in $N_{\uparrow,1}(t) - N_{\uparrow,2}(t)$, etc. For simplicity we will hereafter consider only the case where the initial condition has $N_{\uparrow,1} - N_{\uparrow,2} = J_{\uparrow,1} - J_{\uparrow,2} = N_{\downarrow,1} - N_{\downarrow,2} = J_{\downarrow,1} - J_{\downarrow,2} = \delta N(t=0) \neq 0$, i.e. equal numbers of up and down spin electrons are added at the right Fermi point of one chain. We will follow the dynamics of $\langle \delta N(t \neq 0) \rangle$.

This is a somewhat unfamiliar approach to studying $t_\perp$ so it is worth examining the results for the simple case of free fermions. In that case, the problem is exactly soluble. The requirement that the two chains be prepared in states adapted to $t_\perp = 0$ and in which no Tomonaga bosons are excited but in which $\delta N(t=0) \neq 0$ is easily satisfied by simply taking $n_{1,\sigma}(k) = \Theta(k_F - k + \frac{2\pi}{L}\delta N(t=0))\Theta(k_F + k)$ while $n_{2,\sigma}(k) = \Theta(k_F - k)\Theta(k_F + k)$. Since the free fermion problem is a single particle one every k is independent and independent oscillations occur for the $\delta N(t=0)$ states for which $n_{1,\sigma}(k) - n_{2,\sigma}(k) \neq 0$. The exact result for $\langle \delta N(t) \rangle$ is $\delta N(t=0)\cos(2t_\perp t)$. This is exactly analogous to the $\alpha = 0$ case and clearly corresponds to the interchain hopping being coherent. Given this coherence, it is reasonable to expect the correct eigenstates of the system to be built out of single particle states involving superpositions of the particles on different chains; this should occur even though these states are connected only by $t_\perp$. In fact the exact groundstate in this case is built up with exactly this sort of the creation operators for single particle eigenstates, specifically the symmetric and antisymmetric superpositions of the single particle eigenstates of the individual chains. Because of this, we have

$$\begin{aligned} \langle c_1^\dagger(k) c_2(k) \rangle &= \frac{1}{2}\langle n_S(k) - n_A(k) + c_S^\dagger(k) c_A(k) - c_A^\dagger(k) c_S(k) \rangle \\ &= O(1) \\ &\gg O(t_\perp / E_F) \end{aligned} \tag{35}$$

From this it is clear that the spectral function for the interliquid hopping acting on the ground state also acquires a delta function term with finite weight. The coherence is thus manifest in the nature of the groundstate.

The Fermi liquid case is very similar to the free fermion case, but there are a number of added complexities. To deal with these the natural language to use is that of spectral functions, as it was for the case of the TLS. In a Fermi liquid, the typical one electron spectral function consists, for momenta near the Fermi surface, of a δ-function at energy $\epsilon(k)$ with weight Z and a broad incoherent background of weight $(1-Z)$ which vanishes near $\omega = 0$ at least as fast as ω^2. The typical spectral function for an operator $c_1^\dagger(k) c_2(k)$ then

consists of a δ-function at $\omega = 0$ [43] with weight $Z^2\Theta(k_F^1 - k)\Theta(k - k_F^2)$. The background contains a piece with weight $(1-Z)^2$ vanishing at small ω like ω^5 and a piece with weight $2Z(1-Z)$ which is zero for all $\omega < \epsilon(k) - E_F^1 + E_F^2$ and vanishes linearly with $\omega - \epsilon(k) - E_F^1 + E_F^2$. In a quasiparticle picture, $t_\perp$ would consist primarily of an operator which hopped single quasiparticles k diagonally with matrix element $t_\perp Z$. There would also be terms that removed one quasiparticle from one chain and inserted two quasiparticles and a quasihole in the other while conserving momentum and so on. The first operator results in the δ-function in the spectral function, while the second results in the term which is zero for all $\omega < \epsilon(k) - E_F^1 + E_F^2$ and vanishes quadratically with $\omega - \epsilon(k) - E_F^1 + E_F^2$ and so on. When summed over k the second term vanishes linearly in ω since the coefficient at a given k at small ω is proportional to $(\omega - \epsilon(k) - E_F^1 + E_F^2)^{-2}$. This is fast enough that, for small $t_\perp$, only the delta function term is important because of the vanishing of the other contributions to the spectral function at low frequency. To see this, examine the slowest vanishing contribution, the term quadratic in $\omega - \epsilon(k) - E_F^1 + E_F^2$, which, summed over k, leads to a contribution to the spectral function of $\sum_k c_1^\dagger(k)c_2(k)$ vanishing like ω for $\omega \gg v_F\delta N(t=0)$. The contribution of this term to the probability to have made a transition is

$$\begin{aligned} P_{\text{nextleading}}(t) &\propto \int d\omega\omega \frac{\sin^2(\omega t)}{\omega^2} \\ &\propto \int d\omega\omega^{-1}\sin^2(\omega t) \end{aligned} \tag{36}$$

which is only logarithmically divergent in the infrared at long times, signalling that the operator is only marginally relevant. As one would expect, this can be made arbitrarily smaller than the contribution from the relevant operator. This occurs because the spectral function for the marginal one contains two inverse powers of E_F and thus at long times contributes only

$$P_{\text{marginal}} \sim Lt_\perp^2 \ln[\delta N(t=0)]E_F^{-2} \tag{37}$$

which is much smaller than the contribution from the relevant operator:

$$P_{\text{relevant}} \sim \delta N(t=0)\, t_\perp^2 t^2 \tag{38}$$

For $\omega < v_F\delta N(t=0)$ the leading correction to the single quasiparticle part of the spectral function is finite but this leads again only to a correction that is

$$P_{\text{low}\omega} \sim Lt_\perp^2 t^2 (v_F\delta N(t=0))^2 E_F^{-2} \tag{39}$$

at short times and

$$P_{\text{low}\omega} \sim Lt_\perp^2 t v_F \delta N(t=0) E_F^{-2} \tag{40}$$

at long times. Both are much smaller than the quasiparticle contribution.

In general, an operator whose spectral function vanishes faster than linearly will lead to an infrared convergent contribution to $P(t)$, corresponding directly to the renormalization

group definition of irrelevance. The higher order (in terms of quasiparticles) contributions to ρ are thus irrelevant.

We can therefore neglect the irrelevant and marginal contributions to the spectral function of the single particle hopping operator (though these can certainly generate finite renormalizations they will not change the physics qualitatively) provided that $t_\perp$ is small enough. The contribution to $P(t)$ coming from the delta function in the single particle hopping spectral function then leads to the conclusion that a Fermi liquid behaves, at lowest order, in the same way as non-interacting particles, except that the hopping matrix element $t_\perp$ is renormalized by a factor of Z. Since, in this case, the important hops occur only in one direction, the leading behavior of $\langle \delta N(t) \rangle$ is the same as $P(t)$. Further, since only the quasiparticle contribution is important, we expect that the behavior at higher order will be simple: $\langle \delta N(t) \rangle$ will oscillate with frequency $Zt_\perp$. With modest assumptions about ergodicity it is also clear that, at higher order, $P(t)$ must vanish in the thermodynamic limit after times of order L^{-1} since the system will never return to its original microstate, but it is possible that observable oscillations might be present in $P(t)^{\frac{1}{\tau}}$, signalling coherence. If present, these oscillations will also be at frequency $Zt_\perp$.

These results are entirely consistent with the usual picture for a Fermi liquid where the quasiparticles dominate the physics. In fact in going to higher orders the only difference that appears in the Fermi liquid calculation is the presence of interactions among the quasiparticles. It is also easy to see that if the Landau interaction parameters remain finite, then the effect of the interactions on the response is proportional to the number of quasiparticles involved in the response, so that in the limit of small $t_\perp$ and small $\delta N(t=0)$, the effects of the interactions on $\langle \delta N(t) \rangle$ are negligible and the free particle picture with its coherent oscillations at a renormalized $t_\perp$ obtains. Again, the coherence of the response to $t_\perp$ is consistent with the fact that the groundstate of the coupled Fermi liquids is built up out of quasiparticle creation operators which act on both chains simultaneously, either symmetrically or antisymmetrically, but generating phase coherent superpositions. These phases can only be meaningful if we can observe interference effects between histories in which $t_\perp$ acts.

Notice that in this sense free electrons and Fermi liquids exhibit a heretofore unremarked on *macroscopic quantum coherence*: the total number difference between two chains (or planes) of free electrons is a macroscopic variable which would undergo oscillations, rather than incoherent relaxation, if a finite interchain hopping were suddenly turned on. Viewed in this light it is not surprising that there should exist states in which this macroscopic variable loses its coherence. Rather, it is surprising (though undoubtedly correct for all normal metals) that macroscopic quantum behavior should occur in generic materials. It is interesting that this macroscopic quantum coherence has not previously occasioned some concern in the theory of interacting electronic systems. For example, the above arguments guaranteeing the presence of coherent oscillations in Fermi liquids relied in several places on the quasiparticle structure of the Fermi liquid, which fails totally for interacting fermions in one spatial dimension. We therefore believe that the postulate of previous works, on arrays of chains of interacting fermions coupled by a single particle hopping [35], that the relevance of $t_\perp$ signals a crossover to a three dimensionally coherent Fermi liquid is just that: a postulate. In fact, we will see that the extension to coupled Luttinger liquids of the tools we have used for the TLS problem and coupled Fermi liquids does not support the conclusion that a relevant $t_\perp$ is always a coherent $t_\perp$.

To begin our analysis of coupled Luttinger liquids, we require the spectral function of the single particle hopping operator between otherwise isolated Luttinger liquids. This is easily obtained from the spectral function of the single particle Green's function. The universal features of this function are readily accessible [44,45] and in X we give a simple method for calculating the hopping spectral function from the single particle spectral function. The results of the method when applied to the single electron creation operator are in agreement with those previously obtained [44] and an example is shown in Figure 4. At the level of a linearized dispersion relation, the annihilation operator for momentum k [46] has the same spectral function (when the Fermi energy contribution to the energies is taken out) as the creation operator for momentum $2k_F - k$. The hopping operator's spectral function can be obtained by convolving the spectral function of the individual creation and annihilation operators.

For $\delta N(t=0) = 0$, the spectral function for $\sum_k c_1^\dagger(k)c_2(k)$ is given by $L\ \alpha\omega^{4\alpha}\Lambda^{-(1+4\alpha)}$ where $2\alpha = \frac{1}{4}(K_\rho + K_\rho^{-1} - 2)$ is the anomalous exponent of the single particle Green's function for the case with spin and $2\alpha = \frac{1}{2}(K_\rho + K_\rho^{-1} - 2)$ for the spinless case. Since the spectral function vanishes as $\omega \to 0$ the response to $t_\perp$ is always incoherent for $\delta N(t=0) = 0$. This should not be surprising since for $\delta N(t=0) = 0$ there is no possibility of coherent oscillations in $\langle \delta N(t) \rangle$. It is for this reason that for free particles there would be no response for $\delta N(t=0) = 0$ since their response is entirely coherent. Fermi liquids would have a response but there would be no long time singular behavior due to relevant operators [47], the relevant part of $t_\perp$, the single quasiparticle hopping, having been completely blocked for $\delta N(t=0) = 0$.

Notice that for the Luttinger liquid, the long time incoherent response *is singular* provided that $4\alpha < 1$, despite the fact that coherent hopping is totally blocked. This suggests that the incoherent hopping which was marginal for a Fermi liquid is relevant here and that flows away from the $t_\perp = 0$ fixed point may be dominated by this relevant operator, rather than the relevant operator corresponding to coherent interliquid hopping. If this is the case, then the renormalization group flows should end elsewhere than Fermi liquid theory. In any case, it shows that the failure of the quasiparticle picture has a profound effect here. This is due not only to the anomalous exponent of the Luttinger liquid, but also the destruction of the Fermi surface. No such effect would be present for a model with a single particle Green's function of the form $G(k,\omega) \sim (\omega - vk)^{-1+2\alpha}$.

To see the long time singularity, in addition to $\langle \delta N(t) \rangle$ which at lowest order involves the difference of the spectral function for $\sum_k c_1^\dagger(k)c_2(k)$ and $\sum_k c_2^\dagger(k)c_1(k)$, and therefore vanishes for $\delta N(t=0) = 0$ (as it should):

$$\langle \delta N(t) \rangle = \delta N(t=0) - 4t_\perp^2 \int d\omega \frac{\sin^2(\omega t/2)}{\omega^2} \left(\rho_{1\to 2}(\omega) - \rho_{2\to 1}(\omega)\right) + \dots \tag{41}$$

it is useful to consider the quantity $P(t)$, defined as the probability to remain in the initial state:

$$\begin{aligned} P(t) &= |\langle O| \exp(i \int_0^t dt' H'(t'))|0\rangle| \\ &\sim 1 - 4t_\perp^2 \int d\omega \frac{\sin^2(\omega t/2)}{\omega^2} \left(\rho_{1\to 2}(\omega) + \rho_{2\to 1}(\omega)\right) + \dots \end{aligned} \tag{42}$$

Note that oscillations in δN are the natural signature of coherence and no oscillatory behavior in $P(t)$ is expected in general, however it is useful to talk about both since there may be important effects in $P(t)$ that are obscured in $\delta N(t)$ by the subtraction. The potential for an important effect found at lowest order in $P(t)$ and not visible in the lowest order version of $\langle \delta N(t) \rangle$ is not present in a Fermi liquid since the only relevant terms in that case come from hopping in a single direction and therefore are contained in $\langle \delta N(t) \rangle$.

Let us now procede with a short time expansion analogous to that which we used for the TLS. This should be valid for determining the presence or absence of coherence for exactly the same reason as it was for that problem: the presence or absence of coherence is equivalent to the near degeneracy or non-degeneracy of the states connected to the initial state by $t_\perp$. The short time expansion is capable of revealing such features since it is valid out to precisely the timescale where the initial state has been left behind.

For spinless fermions and finite $\delta N(t=0)$ ($k_F^1 > k_F^2$) the initial spectral function for $\sum_k c_1^\dagger(k) c_2(k)$ is given by (see Sec. IV and X) :

$$\rho_{2\to 1}(\omega) = \Gamma^{-1}(2\alpha)\Gamma^{-1}(2+2\alpha)(2v_S)^{-(1+4\alpha)}\Theta\left(\omega - (E_F^1 - E_F^2) - v_S(k_F^1 - k_F^2)\right) \left(\omega - (E_F^1 - E_F^2) - v_S(k_F^1 - k_F^2)\right)^{1+2\alpha} \left(\omega - (E_F^1 - E_F^2) + v_S(k_F^1 - k_F^2)\right)^{-1+2\alpha} \tag{43}$$

likewise

$$\rho_{1\to 2}(\omega) = \Gamma^{-1}(2\alpha)\Gamma^{-1}(2+2\alpha)(2v_S)^{-(1+4\alpha)}\Theta\left(\omega - (E_F^1 + E_F^2) - v_S(k_F^1 - k_F^2)\right) \left(\omega - (E_F^2 - E_F^1) - v_S(k_F^1 - k_F^2)\right)^{-1+2\alpha} \left(\omega - (E_F^2 - E_F^1) + v_S(k_F^1 - k_F^2)\right)^{1+2\alpha} \tag{44}$$

The short time behavior of $\delta N(t)$ would naively look highly coherent for small $\delta N(t=0)$ due to the subtraction of the two spectral functions and is not that different from what occurs in a Fermi liquid (at least at small α). On the other hand, if we look only at hops in one direction (even the favored one) then the incoherent high energy tail present in the spectral function for $P(t)$ enters and things look much less coherent. $P(t)$ is radically different, even at small α from the Fermi liquids case. In fact for small $\delta N(t=0)$, the incoherent part completely dominates $P(t)$. One way to disentangle these effects is to consider the spectral functions broken down into the contributions coming from individual momenta. First, examine the spectral function for $c_1^\dagger(k)c_2(k)$, which may be obtained by convolving the spectral functions for $c_1^\dagger(k)$ and $c_2(k)$. The spectral function for $c_1^\dagger(k)$ has support for $\omega > E_F^1 + v_S|k - k_F^1|$, where v_S is the sound velocity of the Luttinger liquid. The spectral function behaves at large ω like

$$\rho_{\dagger,1}(\omega\ large) \sim \omega^{-1+2\alpha} \tag{45}$$

and behaves for $\omega \to E_F^1 + v_S|k - k_F^1|$ like

$$\rho_{\dagger,1}(\omega\ small) \sim \left(\omega - (E_F^1 + v_S|k - k_F^1|)\right)^{\alpha - H(k-k_F^1)} \tag{46}$$

where $H(x) = 1$ if $x \geq 0$ and O if $x \leq 0$. The integrated weight is $1 - n_1(k)$. The spectral function for $c_2(k)$ has support for $\omega > -E_F^2 + v_S|k - k_F^2|$, also behaves at large ω like

$$\rho_2(\omega\ large) \sim \omega^{-1+2\alpha} \tag{47}$$

and behaves for $\omega \rightarrow \ -E_F^2 + v_S|k - k_F^2|$ like

$$\rho_2(\omega\ small) \sim \Big(\omega - (v_S|k - k_F^2| - E_F^2)\Big)^{\alpha - H(k_F^2 - k)} \tag{48}$$

The integrated weight is $n_2(k)$.

The convolution has support for $\omega > E_F^1 - E_F^2 + v_S|k - k_F^1| + v_S|k - k_F^2|$ which means that except for $k_F^2 \leq k \leq k_F^1$ the threshold is $E_F^1 - E_F^2 + 2v_S|k - k_F^{avg}|$ where $k_F^{avg} = (k_F^1 + k_F^2)/2$. The behavior as $\omega \rightarrow E_F^1 - E_F^2 + 2v_S|k - k_F^{avg}|$ is

$$\rho_{\dagger,1,2}(\omega\ small) \sim \Big(\omega - E_F^1 - E_F^2 + 2v_S|k - k_F^{avg}|\Big)^{4\alpha} \tag{49}$$

For the case, where $k_F^2 \leq k \leq k_F^1$, the threshold is smallest; there the k independent threshold is given by

$$\begin{aligned} \omega_{min}(k) &= E_F^1 - E_F^2 + v_S(k - k_F^2) - v_S(k - k_F^1) \\ &= E_F^1 - E_F^2 + v_S(k_F^1 - k_F^2) \\ &> 0 \end{aligned} \tag{50}$$

The behavior of the spectral function as $\omega \rightarrow E_F^1 - E_F^2 + v_S(k_F^1 - k_F^2)$ is

$$\rho_{\dagger,1,2}(\omega\ small) \sim \Big(\omega - E_F^1 - E_F^2 + v_S(k_F^1 - k_F^2)\Big)^{1+4\alpha} \tag{51}$$

The positivity of the minimum energy results and the large exponent with which the spectral function vanishes result from the fact that hops in this direction are 'wrong way' hops, that is they increase rather than decrease the initial δN.

The behavior for large ω is given by

$$\rho_{\dagger,1,2}(\omega\ large) \sim \omega^{-1+4\alpha} \tag{52}$$

The integrated weight is $(1 - n_1(k))n_2(k)$.

The spectral function for $c_2^\dagger(k)c_1(k)$ is similar. The spectral function for $c_2^\dagger(k)$ has support for $\omega > E_F^2 + v_S|k - k_F^2|$ behaves at large ω like

$$\rho_{\dagger,2}(\omega\ large) \sim \omega^{-1+2\alpha} \tag{53}$$

and behaves for $\omega \rightarrow E_F^2 + v_S|k - k_F^2|$ like

$$\rho_{\dagger,2}(\omega\ small) \sim \Big(\omega - (E_F^2 + v_S|k - k_F^2|)\Big)^{\alpha - H(k - k_F^2)} \tag{54}$$

The integrated weight is $1 - n_2(k)$. The spectral function for $c_1(k)$ has support for $\omega > -E_F^1 + v_S|k - k_F^1|$, also behaves at large ω like

$$\rho_1(\omega\ large) \sim \omega^{-1+2\alpha} \tag{55}$$

and behaves for $\omega \rightarrow \ -E_F^1 + v_S|k - k_F^1|$ like

$$\rho_1(\omega\ small) \sim \Big(\omega - (v_S|k - k_F^1| - E_F^1)\Big)^{\alpha - H(k_F^1 - k)} \tag{56}$$

The integrated weight is $n_1(k)$.

The convolution has support for $\omega > E_F^2 - E_F^1 + v_S|k - k_F^1| + v_S|k - k_F^2|$ which means that except for $k_F^2 \leq k \leq k_F^1$ the threshold is $E_F^2 - E_F^1 + 2v_S|k - k_F^{avg}|$. The behavior as $\omega \to E_F^2 - E_F^1 + 2v_S|k - k_F^{avg}|$ is

$$\rho_{\dagger,2,1}(\omega\ small) \sim \Big(\omega - E_F^2 - E_F^1 + 2v_S|k - k_F^{avg}|\Big)^{4\alpha} \tag{57}$$

For the case, $k_F^2 \leq k \leq k_F^1$ threshold is $E_F^2 - E_F^1 + v_S(k_F^1 - k_F^2)$, which vanishes for weak interactions and is always smaller than the threshold for hops in the other direction. The behavior as $\omega \to E_F^2 - E_F^1 + v_S(k_F^1 - k_F^2)$ is

$$\rho(\omega\ small) \sim \Big(\omega - (E_F^2 - E_F^1 + v_S(k_F^1 - k_F^2))\Big)^{-1+2\alpha} \tag{58}$$

There is a power law divergence.

The behavior for large ω is given by

$$\rho_{\dagger,2,1}(\omega\ large) \sim \omega^{-1+4\alpha} \tag{59}$$

The total weight in the spectral function is given by $n_1(k)(1 - n_2(k))$.

In the region where $k_F^2 \leq k \leq k_F^1$ a Fermi liquid spectral function would be a delta function at zero frequency, but here there is a power law singularity at a non-zero, negative frequency since $E_F^1 - E_F^2 = (k_F^1 - k_F^2)\frac{v_J + v_N}{2}$ is always larger than than $v_S(k_F^1 - k_F^2) = \sqrt{v_J v_N}(k_F^1 - k_F^2)$ [48]. The essential points are that the singularity is in general not at zero energy and is a power law rather than a delta function.

Notice that when $4\alpha > 1$, none of the spectral functions for the individual momenta in either direction is decreasing for large ω. The high energy behavior is thus incoherent which implies that the response in the $t_\perp \to 0$ limit is always incoherent at every k at short times. The low energy behavior for $\delta N(t = 0) = 0$ is also incoherent. This implies that the proposal of incoherence for $\alpha \geq 1/4$ is self-consistent, since incoherence leads to a vanishing oscillation frequency, ω_{osc}, and therefore the natural $k_F^1 - k_F^2$ to consider is $\omega_{osc}/v_F = 0$, a case where all the spectral functions are in fact incoherent. Notice that for $\alpha < 1/4$ this consistency is not present since even for $k_F^1 - k_F^2 = 0$, some of the spectral functions associated with momenta near to the Fermi surface are coherent [38]. The self-consistency is therefore not trivial and we believe that there must be incoherence for all $\alpha > 1/4$.

This is despite the fact that the low energy form of the spectral function for finite $\delta N(t = 0)$ can have singular behavior (see Eq. 58). The reason for this is that the question of coherence is not one that involves the singularities of the spectral function, but rather comparing the low and high energy contributions to $\int d\omega \rho(2\omega)\frac{\sin^2(\omega t)}{\omega^2}$, as defined previously. From that point of view it is clear that a low energy singularity does not necessarily imply coherence if the high energy behavior is incoherent and the singularity is integrable. Instead, one needs to compare the energy scale at which the spectral functions behavior crosses over between coherent and incoherent to the energy scale which defines the division into high and low energy parts. The appropriate energy scale for that division is approximately given

by the inverse of the time to leave the initial state $\sim t_\perp^R \sim t_\perp^{1/(1-2\alpha)}$. Based on the results for the TLS problem, we expect $\omega_{osc} \sim t_\perp^R$ for small α but for α finite, it is possible that $\omega_{osc} \ll t_\perp^R$ or even $\omega_{osc} = 0$.

It is clear that if $v_F(k_F^1 - k_F^2) << t_\perp^R$, the spectral function differs from the $v_F(k_F^1 - k_F^2) = 0$ case only by corrections at energies low compared to the scale for the high-energy low energy division. Then neither the high nor low energy integrals is greatly changed and the incoherence for $\delta N(t = 0)$ certainly remains. Since the natural choice of $v_F(k_F^1 - k_F^2)$ is ω_{osc} this should apply for $\omega_{osc} < t_\perp^R$, however we also consider the two other possibilities.

In the case $v_F(k_F^1 - k_F^2) >> t_\perp^R$, and strong interactions, the spectral function for hops in the correct direction (the direction which results in a decrease in the initial δN) has a singularity at such a negative frequency (much larger than $t_\perp^R$) that coherence is again ruled out (see Eq. 58, also there are more important effects when spin is included that preclude coherence in this regime). Therefore, the only threatening case is that where $v_F(k_F^1 - k_F^2) \sim t_\perp^R$. In this case, the crossover of the spectral function to its high energy $\omega^{-1+4\alpha}$ behavior occurs at roughly the same frequency as the division into high and low energy pieces and the high energy contribution to transitions out of the initial state is not greatly affected. The effect on the low energy part is difficult to calculate as it depends on the division into high and low energy and on the exact time scale at which one is making the comparison. In general, the velocity inequality produces an $O(1)$ decrease in the low energy part, whereas the integral divergence in the spectral function at low energies produces an $O(1)$ increase in the low energy part and the effect on coherence is ambiguous. In view of this, the self-consistency of incoherence for $\alpha > 1/4$ and the fact that for $\alpha \to 1/4$ we expect $\omega_{osc} \ll t_\perp^R$, it seems unlikely that there exists a second self-consistent solution which is coherent for $\alpha > 1/4$, although we cannot rule this out. We can rule out the possibility that the incoherent phase is eliminated by being pushed all the way to $\alpha = 1/2$, where irrelevance sets in. This cannot occur because, as α approaches $1/2$, the high energy piece is diverging relative to the low energy, and, as we have seen, the initial $\delta N(t = 0)$ makes only an $O(1)$ change in these quantities.

In conclusion, we believe that the behavior at long times will be incoherent for all $\alpha > 1/4$ and, in that case, the ground state should not involve a phase coherent superposition of quasiparticle states, and, more generally, interference effects between histories involving $t_\perp$ should be unobservable. The $\alpha < 1/4$ case is less favorable for coherence, but there are other more complicated reasons for believing that incoherence may extend into that region, as we will now discuss.

A. Dynamics for $\alpha < \frac{1}{4}$

Since, at $\alpha = 1/4$, the spectral function for single particle hopping at any k in any direction is less coherent than the spectral function of the TLS at the onset of incoherence [38], coherence in the dynamics of the fermion model would have to result from an interaction of electrons hopped at different k's. While we can not explicitly rule this out without a full solution of the strongly interacting problem at strong coupling and finite $t_\perp$, the idea seems implausible since such interactions are irrelevant in Fermi liquid theory and, in so far as they generate finite renormalizations there, they are hostile to coherence. Certainly, the states

connected *to the appropriate initial state for small* $\delta N(t=0)$ by the single particle hopping operator are effectively nondegenerate, and all that is required for incoherence is that this nondegeneracy is typical. From this point of view the possibility that interactions somehow restore coherence at higher orders seems highly unlikely and we know of no examples of this behavior in soluble models. Given this, we take the incoherence of the dynamics for $\alpha > 1/4$ for granted and pass now to a discussion of the more complicated case of smaller α. Readers not interested in the more technical arguments for incoherent dynamics for $\alpha < \frac{1}{4}$ should skip to section V.

Consider the case where there is only one extra particle which may oscillate coherently back and forth, i.e. $\delta N(t=0) = 1$. In this case, for the Fermi liquid, the arguments for general δN show that one expects various irrelevant effects plus coherent oscillations of period approximately equal to $(Zt_\perp)^{-1}$, however the Luttinger liquid case is quite different. Let us consider the lowest order contributions to $P(t)$ and $\langle \delta N(t) \rangle$. $P(t)$ is the simpler to treat since it is finite for $\delta N = 0$. So long as $\delta N(t=0) \ll t_\perp / v_F$, the $\delta N(t=0) = 0$ results can be used for $P(t)$ so that the lowest order result is

$$\begin{aligned} P(t) &= 1 - 4t_\perp^2 \int d\omega \frac{\sin^2(\omega t)/2}{\omega^2} L\ \alpha \omega^{4\alpha} \Lambda^{-(1+4\alpha)} \\ &\sim 1 - \frac{\alpha}{1-4\alpha} 2^{1+4\alpha} t_\perp^2 L t^{1-4\alpha} \end{aligned} \tag{60}$$

For $\delta N(t=0) = 1$, there is only one extra particle which may oscillate coherently back and forth. The total weight in the spectral function that enters into computing $\langle \delta N(t) \rangle$ is $n_1(k) - n_2(k)$, and therefore, since the extra added electron is spread out over a range of k's, the spectral function for $\langle \delta N(t) \rangle$ then has support over many momenta. In frequency things are more localized. Since the spectral function contributing to $\langle \delta N(t) \rangle$ involves the difference of the spectral function of $c_2^\dagger c_1$ and $c_1^\dagger c_2$, it is sharply peaked at the lowest possible energies at a given k, decaying for $\omega \gg v_F \frac{2\pi}{L} \delta N$ like $\omega^{-2+4\alpha}$. As a result, the spectral function for $\langle \delta N(t) \rangle$ can be approximated by a delta function for small δN and $\alpha < 1/4$. There is a complication due to the fact that the weight must be reduced to reflect the fact that the amount of weight in the low energy part is not

$$\begin{aligned} n_1(k) - n_2(k) &\sim \frac{2\pi}{L} \left(-\frac{\partial n}{\partial k} \right) \\ &\sim |k - k_F^{avg}|^{-1+2\alpha} \end{aligned} \tag{61}$$

This arises because the $\omega^{-2+4\alpha}$ behavior is valid only for energies much smaller than the cutoff (but much larger than $v_F \delta N$, of course). There is weight pushed out past the cutoff on the single particle Green's function by convolving two of them so that in fact

$$\begin{aligned} (n_1(k) - n_2(k))_{\text{low energy}} &\sim O(\frac{\alpha}{L}) \left(-\frac{\partial}{\partial k} |k - k_F^{avg}|^{2\alpha} \int_{|k-k_F^{avg}|}^{2|k-k_F^{avg}|} d\omega (\omega - |k - k_F^{avg}|)^{-1+2\alpha} \right) \\ &\sim O(\frac{\alpha}{L}) |k - k_F^{avg}|^{-1+4\alpha} \end{aligned} \tag{62}$$

We then find:

$$\langle \delta N(t) \rangle \sim \delta N(t=0) \left(1 - t_\perp^2 O(\frac{\alpha}{L}) \int d\omega \sum_k \frac{\sin^2(\omega t)}{\omega^2} \delta(\omega - 2v_S|k - k_F^{avg}|)|k - k_F^{avg}|^{-1+4\alpha} \right) \quad (63)$$
$$\sim \delta N(t=0) \left(1 - t_\perp^2 O(\alpha) \int_0^\infty dx \frac{\sin^2(v_S x t))}{v_S^2 x^2} x^{-1+4\alpha} \right)$$
$$\sim \delta N(t=0) \left(1 - O(1) t_\perp^2 t^{2-4\alpha} \right)$$

The time dependence of $\langle \delta N(t) \rangle$ is thus intrinsically coherent at lowest order for $\alpha < 1/4$. This is in agreement with the finding that some of the momentum channels contributing to $P(t)$ had coherent spectral functions for $\alpha < 1/4$.

We then have clear indications of coherence in δN for $\alpha < 1/4$, however, the number of transitions contributing to $P(t)$ outnumber by a factor of $L/\delta N$ those contributing to $\langle \delta N(t) \rangle$. This factor is thermodynamically large for $\delta N = 1$ and still very large for $\delta N \sim L t_\perp / v_F$. The question then becomes whether the coherence in $\delta N(t)$ can be believed in the presence of $P(t)$'s behavior.

If we adopt the maximally collective point of view and ask only if the system has left its initial state before the extra particle has moved between the two chains, then it is clear that, in the thermodynamic limit ($L \to \infty$) for finite $\delta N(t=0)$, the system always changes state first. The incoherent $P(t)$ might then be expected to dominate the coherent $\delta N(t)$. This seems extreme, however, since it is clear that there must be some interaction between the hops associated with $P(t)$ and those associated with $\delta N(t)$ for the coherence of $\delta N(t)$ to be spoiled and the maximally collective assumption amounts to assuming that every electron participating in an $P(t)$-type hop will interact with every electron which would have participated in a $\delta N(t)$-type hop. A more reasonable approximation might be to take the other extremal point of view, that those hops contributing to $P(t)$ that occur at a given wavevector, k, affect only the contribution to $\delta N(t)$ associated with that same wavevector and the same direction. This amounts to allowing the electrons to interact only "statistically", and should be a minimal assumption. Clearly, if there has just been a transition induced by $c_1^\dagger(k) c_2(k)$ to a high energy state, no transition induced by $c_1^\dagger(k) c_2(k)$ to a low energy state (or any other state for that matter) is possible until both the hole and particle have scattered. This is similar again to the TLS problem where the spin can not flip in the same direction twice and thus high energy flips completely destroy the coherence once their number becomes of order one per low energy flip. In our case, for the incoherent hops to plausibly destroy the coherence, we need that probability to have undergone an $P(t)$-type hop (at every wavevector k for those k's that contribute to $\delta N(t)$) be of order unity or larger for times of order the inverse of the putative oscillation time (recall that the expected oscillation time is of order $t_\perp^{1/(4\alpha-1)}$ or much longer). For $\delta N \ll L$, we can use the spectral functions for $\delta N = 0$ in computing $P(t)$. For the most important momenta (those near the Fermi surface) we find contributions to $P(t)$

$$\delta P(t) \sim 4\alpha t_\perp^2 \int d\omega \sin^2(\omega t/2) \omega^{-1+4\alpha} \Lambda^{-4\alpha} \quad (64)$$
$$\sim \frac{1}{4} t_\perp^2 t^{2-4\alpha} \Lambda^{-4\alpha}$$
$$\sim 1$$

We see that for one extra particle, there is no compulsion to believe in coherence for any α! However, it is not clear either that the case of one extra particle settles the question or that our counting all $P(t)$ type processes as incoherent is entirely fair. If we instead take the incoherent processes to be those involving energies larger than $t_\perp^R$ [49], we find an effect of order $O(\frac{\alpha}{1-4\alpha})$ [50]. The effect is essentially identical to what occurs in the TLS if one took the TLS spectral function for $(\alpha_{TLS} = 2\alpha_{fermion})$. This destroys coherence for $\alpha > 1/4$, as we argued it should earlier when we compared the spectral function of $c_1^\dagger(k)c_2(k)$ to that of the TLS. This seems more reasonable since it is in agreement with our findings for the spectral functions for individual momenta, and, as we shall see, is also in agreement with our results for larger δN.

Let us now consider the effect on $P(t)$ of the natural choice for $\langle \delta N(t=0)$: $\langle \delta N(t=0)\rangle \sim t_\perp^R/v_S$. For this case, the spectral function for $\delta N(t)$ essentially contains all of the weight in the correct direction which is also low energy, i.e. lower than $t_\perp^R$, paralleling the definition we introduced in the TLS discussion. In this case our crude division of processes into high and low energy at the energy scale $t_\perp^R/v_S$ is equivalent to the division into $\delta N(t)$ and $P(t) - \delta N(t) + \delta N(t=0)$. This further motivates the study of $\delta N(t=0)$ of this size. Now, the spectral function for $P(T)$ is essentially modified because of the substantial $\delta N(t=0)$. In fact, is easy to see that there is essentially no weight in $P(t)$ for energies below $v_F \delta N(t=0)$ so that the above contribution from $P(t)$ type hops is modified to

$$\begin{aligned} \delta P(t) &\sim 4\alpha t_\perp^2 \int_\lambda^\infty d\omega \frac{\sin^2(\omega t/2)}{\omega}^{-1+4\alpha} \Lambda^{-4\alpha} \\ &\sim \frac{\alpha}{1-2\alpha} t_\perp^2 \lambda^{-2+4\alpha} \Lambda^{-4\alpha} \\ \delta P(t=\lambda^{-1}) &\sim \frac{\alpha}{1-2\alpha} \end{aligned} \tag{65}$$

where $\lambda = t_\perp^R$. This is then in complete agreement with our earlier discussions.

We see that, by itself, the minimal statistical interaction is equivalent to looking at the individual momentum channels and reproduces our earlier criterion for incoherent dynamics. Let us now consider the non-statistical interaction of the hopped electrons with each other. The first important point to consider is that, in a Luttinger liquid, the four point function does not factorize into products of two point functions, signaling that inserting two electrons has a very different effect than inserting one. This is due to the strong interactions between the inserted electrons *even as they approach the Fermi surface*. If the two added electrons are very much farther from each other in space time than they are from the added holes, then the four point function will approach the product of two point functions; otherwise, the two added particles should be regarded as interacting strongly with each other. In our calculation, the creation and annihilation operators are generally separated by times of order $1/t_\perp^R$, so that if an additional hop has occurred closer than $v_S/t_\perp^R$, that hopped electron will interact strongly and we can expect an $O(1)$ scattering effect and the loss of coherence in that way. Note that this effect is completely absent in Fermi liquids since the four point function always factorizes in the low energy limit where quasiparticle interactions are irrelevant. The density of hopped electrons from the bare incoherent effects at time $1/t_\perp^R$ is roughly given by

$$\rho_{hops} \sim P(t = 1/t_\perp^R)/L \sim \frac{\alpha}{1-4\alpha} t_\perp^R \tag{66}$$

so that in general the chance that an electron attempting to hop coherently will be scattered before the amplitude can be built up to order one is $O(\frac{\alpha}{1-4\alpha})$ [50], of the same order of importance as the statistical interaction, so that we have here an entirely separate source of incoherence from that in the TLS problem. Further, that second source is as strong as the first. It is this additional effect which of course makes our problem a truly many body one and keeps it from being rigorously tractable via any of the known approaches to the TLS. We believe that it is safe to conclude that the critical α will be pushed substantially lower by the combination of the two effects.

There are additional reasons unrelated to $P(t)$ for believing in a lower critical α which appear when macroscopic particle number inequalities are considered. As we make $\delta N(t=0)$ larger, effects related to the plethora of velocities in Luttinger liquids set in. Consider again the case: $\langle \delta N(t=0) \rangle \gtrsim t_\perp^R / v_S$. In this range of $\delta N(t=0)$, several factors relating to velocities in Luttinger liquids neglected in the previous discussion become important. Recall that, in a Luttinger liquid, the velocities for current and charge excitations are in general unequal ($v_N \neq v_S \neq v_J$), implying that the shift of the Fermi energies need not exactly cancel the energy of the bosons associated with hops occurring at momenta between the two k_F's in the correct direction. We found in Eq. 58 that:

$$\rho(\omega \ small) \sim \Big(\omega - (E_F^2 - E_F^1 + v_S(k_F^1 - k_F^2))\Big)^{-1+2\alpha} \tag{67}$$

There is no analog of this in a Fermi liquid where $\epsilon(k)$ is by definition the same in the two identical liquids and, as we suggested earlier, we expect potentially important effects on coherence. In particular, coherence will be substantially disrupted if we make $\delta N(t = 0) \gg t_\perp^R (v_S/|v_N - v_J|^2)$, since this shifts the singularity in the spectral function away from zero energy by an amount much larger than $t_\perp^R$. The shift of the spectral weight due to the inequality of v_N and v_J away from zero energy is an order $O(\alpha)$ effect since $v_N - v_J = O(\sqrt{\alpha})$ while the energy mismatch is of order $O((v_N - v_J)^2)$, and can therefore be considerable at finite α, substantially reducing the critical α required for incoherent dynamics.

An even more substantial velocity inequality occurs between the spin and charge excitation velocities for Luttinger liquids with spin. This spin-charge separation also diminishes coherence as we make $\delta N(t=0)$ larger since the width of the spectral function associated with the electron far from the Fermi surface grows linearly with distance. This drastically changes the spectral function for hopping in the case of finite $\delta N(t=0)$, completely eliminating the divergence in $\rho_{2\to 1}$, replacing it with a threshold singularity at energy $v_\sigma(k_F^1 - k_F^2) - (E_F^2 - E_F^1)$ of the form

$$\rho_{2->1}(k) = \Big(\omega - v_\sigma(k_F^1 - k_F^2) - E_F^2 - E_F^1\Big)^{2\alpha} \tag{68}$$

for hops at momenta, k, between k_F^1 and k_F^2. In addition to the elimination of the divergence, its placement is now at an energy which is very different from zero, particularly if v_σ is small.

There are additional singularities associated with other energies, but no divergences and the physical effect is really to spread all of the weight from the divergence in the spinless case out over a region of size $(v_\sigma - v_\rho)(k_F^1 - k_F^2)$, drastically reducing the possibility for coherence.

The effect resembles introducing an effective lifetime proportional to $\delta N(t=0)$ and severely restricts the possibilities for coherence for initial conditions with large $\delta N(t=0)$.

The effects due to the spin-charge separation should be large for the large U Hubbard model where $v_\sigma \to \frac{t_\parallel^2}{U} \sim 0$, while v_ρ remains finite. For small U where $v_\rho - v_\sigma = O(U)$ the effect is $O(U)$ or, since $\alpha = O(U^2)$, $O(\sqrt{\alpha})$. We again expect that the α required for incoherent dynamics will be substantially reduced by the inclusion of these velocity inequality effects, in effect allowing us to consider only the case $\delta N(t=0) \lesssim t_\perp^R / v_S$. For that case, there is a significant statistical and non-statistical interaction of the high energy hops contributing to $P(t)$ on the hops contributing to $\delta N(t)$ through the same momentum channel. In fact, as we saw, both effects are little changed from the single extra particle case and remain $O(\alpha/(1-4\alpha))$, so that the velocity inequality effects are truly additional to our earlier sources of incoherence.

Notice that, for the case of coupled Hubbard models as $U \to 0$, the velocity inequalities become arbitrarily small, as does α, and for any partition into high and low energy processes, the low energy will dominate. The scattering from incoherently hopped electrons, as estimated above, will vanish as well. Together these things would imply a critical α below which hopping would be coherent even in the $t_\perp \to 0$ limit, however, the arguments in favor of this are not entirely conclusive. We did find suggestions of incoherence for one additional particle for any α due to interparticle interactions. Also, in the limit as $t_\perp \to 0$ the incoherent hops have arbitrarily long time to scatter the coherent hops so that if this scattering is enhanced over our estimate by any divergent amount in the long time limit then incoherence should result as $t_\perp \to 0$. Our original work [6] claimed a logarithmic correction so that the critical U for the coherence-incoherence transition was zero for $t_\perp \to 0$ and the $t_\perp$ required for coherence went like $\exp(const\ U^{-5})$. We have not been able to make the arguments put forward there compelling, so we regard the question of the existence of a critical interaction strength as $t_\perp \to 0$ as open, but favor a finite interaction strength as most likely necessary for incoherence. It seems clear that, since U is a marginal operator while $t_\perp$ is relevant, any phase boundary passing through $U=0$, $t_\perp = 0$ would have to have the form $t_\perp \sim \exp(const\ U^{-x})$. This would involve a logarithmic enhancement of the interaction at long times of the coherent hops by the incoherent ones and the question is therefore rather subtle.

In any case, we have given strong arguments for the existence of incoherence for $\alpha > 1/4$ and demonstrated three factors which should result in incoherence for α substantially lower values of α.

B. Connection to Green's Function Approaches

Hints of the incoherence we are proposing can be seen in more conventional calculations. The interpretation of these calculations is complicated since they are not really intended for studying coherence effects, but in light of what we have learned there are a number of suggestive results. Consider as an example the calculation of the Green's function for a Luttinger liquid of spinless fermions with vertex corrections neglected, as in Wen's work [35] (there have been similar calculations by a number of others [35] and what we have to say

applies, in general, to all of them - we focus on one for the sake of clarity). In that work, Wen studied how the Green's function

$$G_0(k,\omega) = (v^2k^2 - \omega^2)^\alpha(\omega - vk)^{-1} \tag{69}$$

is modified, if vertex corrections are neglected and infinitely many chains are coupled. Here k is momentum along the one-dimensional chains measured from the right Fermi point.

It is easy to see that the neglect of vertex corrections amounts to incorporating $t_\perp(k_\perp)$ as an energy independent self-energy and one expects:

$$G^{-1}(k,k_\perp,\omega) = G_0^{-1}(k,\omega) - t_\perp(k_\perp) \tag{70}$$

Wen actually proposed the form for k near the right Fermi point:

$$G(k,k_\perp,\omega) = \frac{(v^2k^2 - \omega^2)^\alpha}{(\omega - vk) - t_\perp(k_\perp)(v^2k^2 - \omega^2)^\alpha} \tag{71}$$

where we have set the dimensionful high energy cut-off to 1. This is what one expects, from the incorporation of $t_\perp$ as a self-energy. The expression is, however, incomplete in that no discussion of analytic properties of the unperturbed or the perturbed Green's function is given. All the singularities in G_0 are for real ω and must be moved off the real axis. When this is done with care we will see that rather interesting physics results.

Putting aside for the moment the question of analytic properties, Wen found that a zero of the denominator for $\omega = 0$ occurs when

$$vk^{1-2\alpha} + t_\perp(k_\perp) = 0 \tag{72}$$

which has no solution for $\alpha > 1/2$ for small $t_\perp$ and k. As Wen correctly pointed out, this signals the renormalization group irrelevance of $t_\perp$ in this regime.

For $\alpha < 1/2$, he suggested that the solutions may be interpreted as specifying the position of a two dimensional Fermi surface. If the hopping is relevant, then the conclusion of Wen and various others [35] was that a crossover to behavior with coherence in all directions should occur. The transverse bandwidth would be of order the scale at which the splitting was comparable in magnitude to the bare energy, vk, leading to $t_\perp^R \sim (t_\perp)^{\frac{1}{1-2\alpha}}$ [51]. We agree with this interpretation for sufficiently small α for spinless Luttinger liquids (although even here a note of caution should be sounded), however, if we examine the behavior close to the Fermi surface as $\alpha \to 1/4$ there is rather interesting behavior.

First, recall that singularities of the Green's function, particularly poles, have sensible physical interpretations only in the second and fourth quadrants of the complex ω plane. Since, for $k \neq 0$, the Green's function has two branch cut singularities, one for each sign of ω, these originate just off the real axis in the second and fourth quadrants. Further, G_0 must be real for $-vk < \omega < vk$ since in that region no on-shell decay of an injected fermion is possible. This implies that the phase of G_0 for $\omega > vk$ should be given by $-\alpha\pi$, and by $-\pi - \alpha\pi$ for $\omega < -vk$. Recalling this, let us look at the effect on the Green's function at the right Fermi point, $k = 0$, when $t_\perp$ is turned on. Consider $t_\perp(k_\perp) > 0$, then the pole for $\alpha = 0$ is at $t_\perp(k_\perp)$. If we now turn on α, the pole must shift into the fourth quadrant since the pole equation is

$$0 = G_0^{-1}(k,\omega) - t_\perp(k_\perp) \tag{73}$$

and $G_0^{-1}(k,\omega)$ has a phase which is $\alpha\pi$ on the real axis for $\omega > 0$ and decreases as we move down into that quadrant. Moving off the axis into the fourth quadrant an angle Θ changes the phase to $\alpha\pi - (1-2\alpha)\Theta$ and it is again possible to have a pole if

$$\Theta = \pi\frac{\alpha}{1-2\alpha} \tag{74}$$

For small, α this pole can be sensibly interpreted as a weakly damped quasiparticle pole with the corresponding quasiparicle state unoccupied, as it should be. However, for $\alpha \to 1/4$, $\Theta \to \pi/2$ and the "quasiparticle energy" of the pole, given by the real part of ω, becomes smaller and smaller. For $\alpha = 1/4$, $\Theta = \pi/2$ and the real part of the quasiparticle energy is not changed at all. The quasiparticle Fermi surface, determined by the zero crossings of the real part of the pole frequency, is also not moved by $t_\perp$.

For $\alpha > 1/4$, the pole moves out of the fourth quadrant and there is no sensible physical interpretation of the pole. As a quasiparticle pole, it would be signalling an unoccupied, negative energy quasiparticle state. Note that, while this is suggestive of an instability, the spectral function in this approximation remains positive definite for $\alpha > 1/4$ and we know of no argument which rigorously demonstrates an instability setting in for $\alpha > 1/4$. We do, however, regard it as significant that the pole which occurs for $\alpha = 1/4$ has a purely imaginary frequency, entirely in keeping with the idea that $t_\perp$ is acting incoherently at this value of α.

For a negative $t_\perp$, an exactly parallel scenario involving the second, instead of the fourth quadrant, would have resulted. In both cases, for $\alpha > 1/4$, there is no physically sensible interpretation of the pole obtained by the incorporation of $t_\perp$ as a self-energy, and the results are extremely suggestive of incoherence.

The effect is very closely analogous to the behavior of the Laplace transform of $P(t)$ found in [31] at the onset of incoherence. A similar analogy between the locations of the poles of the single particle Green's function approximated in this way and the Laplace transform of $P(t)$ in the TLS problem was noted in [52].

If we now move away from the Fermi surface, are there physically sensible poles in G? First, note that the arguments for the poles we have considered for $k = 0$ will apply essentially without modification to all $k \ll t_\perp^R$ so, for these poles, we must consider k's that are at least $O(t_\perp^R)$. Our approach is to begin for $\alpha < 1/4$ and track the poles as we increase k. Then we will see if an increase in α past $1/4$ still allows sensible poles at finite k. We consider both signs of $t_\perp$ but only positive k since negative k is essentially identical for the opposite sign of $t_\perp$.

Consider first the case $t_\perp > 0$. As we move some distance away from the Fermi surface, the singularity at $-vk$ becomes more distant and its effect on the phase less important. It now becomes possible to circle the singularity at vk without moving appreciably with respect to the singularity at $-vk$. Therefore the phase of $G_0^{-1}(k,\omega)$ varies like

$$\text{phase} = \alpha\pi - (1-\alpha)\Theta \tag{75}$$

where Θ is measured downward from the real $\omega > vk$ half-line. Because of this change, we can reach larger α without the pole being forced into unphysical regions. In fact, it is

possible to attain a phase of $(2\alpha - 1)\pi$ for $k \gg t_\perp^R$ by going around the singularity at vk. Under these circumstances there is an allowed pole for $\alpha < 1/2$ at an angle given by:

$$\Theta = \frac{\alpha}{1-\alpha}\pi \tag{76}$$

For $\alpha > 1/3$, the behavior of this pole is rather bizarre since then $\Theta > \pi/2$ is required. In that case, the addition of a positive real self energy shifts the singularity to a complex energy with a real part less than vk. If we were to add in spin-charge separation and then turn on α then the behavior of the pole is even more bizarre. In that case, the phase of the Green's function changes from $-\pi/2$ to $\pi/2 - 2\alpha\pi$ as ω circles the head of the branch cut at $v_\rho k$. For $\alpha > 1/6$, the pole lies at an energy whose real part is shifted lower than $v_\rho k$, and for $\alpha > 1/4$, it is not clear that there is a pole at all. The phase angle is less than zero after circling $\omega = v_\rho k$, although it reaches the positive value (for $\alpha < 1/2$) $\pi(1-2\alpha)$ as we move to $\omega = v_\sigma k$. Since the phase part of the pole equation requires an ω which is a finite distance away from $v_\rho k$, the Green's function is in general finite where the phase is correct and it is not clear that the modulus aspect of the pole equation is satisfiable. In fact, for large k it is not, while for sufficiently small k, the Green's function can be made large enough and eventually the pole equation will be satisfiable. In this case, there is no physically sensible solution for large enough k and $\alpha > 1/4$, as there was none for small k and $\alpha > 1/4$. These peculiarities probably arise because of the uncontrolled nature of the diagrammatic calculation. Also, without an entirely correct solution near $k = 0$, the analytic properties of G far from $k = 0$ cannot be determined reliably since they depend on the accessible decay channels near $k = 0$. At the level of approximation we are working at, all we can say is that there are definite indications of incoherence for $\alpha > 1/4$ and that such indications of coherence as there are for $\alpha > 1/4$ are very dubious, particularly in spin charge separated models.

Now, let us follow the pole for $t_\perp < 0$ as we increase k. For $\alpha < 1/4$ this pole lay in the second quadrant and the phase requirement upon increasing k will push it first further off the axis into this quadrant and then towards the first quadrant, eventually it will cross over to the first quadrant and a physical interpretation in terms of a quasiparticle pole is impossible. We can calculate at what k this occurs and find

$$k_{\text{critical}} = \frac{1}{v} t_\perp^{1/(1-2\alpha)} \cos(2\pi\alpha) \tag{77}$$

(recall that we are working in units where the high energy cut-off is 1). At this point the pole lies at its last allowed ω:

$$\omega_{\text{critical}} = i\, t_\perp^{1/(1-2\alpha)} \sin(2\pi\alpha) \tag{78}$$

We see that there are allowed k's only for $\alpha < 1/4$ and that k_{critical} is monotonically decreasing with α, so that the pole is always unphysical for $\alpha > 1/4$. Notice also that the correct behavior for $\alpha = 0$ is obtained for this pole which then moves along the real axis and crosses through the origin for $t_\perp = vk$.

In addition to this pole, a new pole always appears for negative $t_\perp(k_\perp)$ for any finite k. For small enough k, the pole sits at

$$\omega \sim -vk + \left(\frac{2vk}{t_\perp}\right)^{1/\alpha} (2vk)^{-1} \tag{79}$$

where the equation is valid as long as the distance of the pole from $-vk$ is much less than vk. The appearance of this pole is not simultaneous with the disappearance of the other pole, except when $\alpha = 1/4$. The new pole moves to some ω very close to but less than vk as we increase k, always remaining on the real axis.

This pole must occur because the Green's function is always real on this region of the real ω axis in this approximation and so a pole is guaranteed for $\alpha > 1/2$ and several authors regard this as important (*e.g.*, Boies *et al.* [35]). The appearance of this undamped pole is, however, an artifact. The Green's function is purely real in this region because there is nothing for a fermion at this momentum and energy to decay into in the unperturbed model. However, in the perturbed model, at this level of approximation, all of the other electron states have been pulled down to lower energies also and there are many accessible decay channels. These are neglected by the neglect of vertex corrections in this approximation. The interpretation of this pole as physical is therefore highly dubious. This is especially true in light of the pole's origin. It not only appears discontinuously as we increase k from zero, but if we take the limit $\alpha \to 0$ at some finite k for which $2vk < t_\perp$, the weight in this pole vanishes rapidly and, moreover, the pole does not approach the position of the pole for the $\alpha = 0$ case (it approaches $-vk$ instead). Conversely, the pole which is connected continuously to the $k = 0$ pole approaches the correct position.

In addition to this strange behavior for small k, the pole on the real axis has very strange properties in the $t_\perp^R \ll v_\rho k$ limit for spin charge separated models. For $\alpha > 1/4$, it should no longer exist since the unperturbed Green's function does not diverge as $v_s k$ is approached. For $\alpha < 1/4$, the pole lies just below $v_\sigma k$, at an energy essentially unrelated to the charge velocity of the model and the original hopping integral in the chain direction. Because of this, in the large U limit of the Hubbard model, where the spin velocity vanishes, the pole is completely dispersionless along the chains. The "quasiparticles" defined by this pole would have zero bandwidth in the direction of large hopping, but propagate coherently with finite bandwidth in the direction of small hopping!

Even in the spinless case, one should also be cautious in interpreting this pole because, as we have seen, it cannot be obtained, for any finite α, by a continuous deformation from poles occurring for small k and $\alpha = 0$. The other poles we have discussed, which do connect up to that limit in a sensible manner, at least in the calculable spinless case, become unphysical immediately after developing a purely imaginary frequency. This occurs for sufficiently large k even for $\alpha < 1/4$. Only for $\alpha = 0$ does the imaginary frequency not arise, since, in this case, the all important branch cuts are absent.

In summary, we see that in this approximation: (1) there no physical poles for $k = 0$ for $\alpha > 1/4$, (2) the last physically interpretable poles occur with purely imaginary frequencies (3) the undamped pole that appears for finite k has a number of unphysical features, (4) for $\mathrm{sgn}(t_\perp) \neq \mathrm{sgn}(k)$ there is no physically meaningful pole for $\alpha > 1/4$, (5) for $\alpha < 1/4$ and $\mathrm{sgn}(t_\perp) \neq \mathrm{sgn}(k)$, the last physically sensible pole again lies at a purely imaginary frequency, (6) the physical pole for $\mathrm{sgn}(t_\perp) = \mathrm{sgn}(k)$ and $\alpha > 1/3$ (1/6 for the spin-charge separated case) lies to the wrong side of $v_\rho k$, and finally (7) this pole is not even present for large k in spin-charge separated models with $\alpha > 1/4$. Taken together, these observations clearly

demonstrate that there is no support for coherence beyond $\alpha = 1/4$ in the Green's function approach. The resummation leading to the Green's function examined is an uncontrolled approximation and should be interpreted with caution, but in so far as it is believable at all, it seems to support the proposal of incoherent interliquid hopping for α's well below the $1/2$ required for renormalization group irrelevance.

It is worth noting that many of the changes in the singularities of G as we move away from the Fermi surface parallel what occurred with the spectral functions in our calculation of $\langle \delta N(t) \rangle$ when we moved away from $\delta N(t=0) = 0$, although the results for the spectral functions are in general more sensible. Recall that a singularity appeared for $\delta N(t=0) \neq 0$ that naively suggested coherence, at least for intermediate δN. The dynamical calculation made it clear that the increase in coherence was not as dramatic as naive exponent counting would suggest, since coherence depended on the properties of the spectral function integrated out to an energy $\sim t_\perp^R$. Incoherence remained a self-consistent possibility for all $\alpha > 1/4$ (at least). In both the Green's function calculation and the $\langle \delta N(t) \rangle$ calculation, the introduction of spin-charge separation effects is clearly important for considering momenta away from k_F or, equivalently, large $\delta N(t=0)$, and it is clearly detrimental to coherence, removing many of the naively coherent behaviors for $k \neq 0$ and $\delta N(t=0) \neq 0$, respectively.

In summary, guided by our results for the behavior of δN, we have been able to identify many strange features of the Green's function approach to the coupled chains problem. What at first sight appear to be peculiarities in that approach are, in fact, fully consistent with our results for incoherence from our earlier calculation. In both cases, coherence for $\alpha > 1/4$ requires a leap of faith that the calculations do not support. Note that the Green's function calculation neglects vertex corrections and, likewise, our dynamical calculation works to lowest order and therefore also neglects the effects of interactions among the hopping fermions. However, we know of no reason to believe that these effects can somehow magically restore coherence to the interchain hopping. In fact, as we argued before they should substantially decrease it since fermions which hop coherently may interact with those hopping incoherently. This may lead to a two particle instability but that is by no means clear since it is not the case that the single particle hopping is renormalizing to zero for $\alpha < 1/2$ (see Sec. VI).

Which approach to studying incoherence is preferable might therefore appear to be a matter of taste. In general, we believe that the dynamical calculation based upon δN is more appropriate for settling questions of coherence. The primary reason for this is that the dynamical calculation is specifically aimed at the question of coherence, being closely analogous to the calculation done for the TLS problem. This carries with it the advantage that what is being calculated is the outcome of a Gedanken experiment and is in principle physically observable, whereas the single particle Green's function is not. To see why this is advantageous consider the fact that the single particle Green's function *can never be correctly obtained directly from zero temperature perturbation theory when the perturbation produces a Fermi surface shift* [53]. The physical reason for this is that Fermi surface shifts, such as those predicted in the coherence regime, are not perturbatively accessible since they involve the rearrangement of thermodynamically many fermions. Moreover, for perturbations which conserve $n_k^1 + n_k^2$, as $t_\perp$ does, there can be no change at *any* order in perturbation theory in $n_k^1 + n_k^2$. A split Fermi surface clearly requires such a change. On the other hand, the dynamical question we have asked naturally involves initial states from the $t_\perp = 0$ problem,

which we understand, and is in principle answerable from a direct calculation. We need not assume in advance what form $n_k^1 + n_k^2$ will have since our choice of initial conditions explicitly allows for coherence (but does not force it, unlike a perturbative calculation *beginning* from a split Fermi surface starting point [35]). In contrast, as we have seen above, the lowest order approximation ($\Sigma = t_\perp$) to the self energy is generally an uncontrolled approximation, leading to a Green's function with properties generally difficult to interpret, and often quite bizarre.

It is worth noting that, for the case of infinitely many coupled chains the behavior of $n(k_x, k_y)$, which is not identically conserved, has been studied at lowest order in perturbation theory by Castellani *et al.* [35]. They find a shift of $n(k_x, k_y)$ which at lowest order is proportional to $\cos(k_y)|k_x^F - k_x|^{-1+4\alpha}$ and interpret this as signaling the instability of the Luttinger liquid. Since they work only to lowest order, the effects of vertex corrections, or equivalently the effects of interactions between hopped fermions, are neglected. Taking this into account, their results are fully consistent with ours, since they find a qualitative change in the behavior exactly for $\alpha = 1/4$. Their instability is equivalent to the renormalization group relevance of $t_\perp$, with which we agree, and the change in behavior at $\alpha = 1/4$ would be perfectly compatible with flows leading off to an incoherent but finite $t_\perp$ fixed point, rather than a coherent and finite $t_\perp$ fixed point.

To conclude, there is strong evidence for incoherence in Green's function calculations, consistent with our dynamical calculation; however, the evidence is more difficult to interpret in the former, and the dynamical calculation is more suited to the study of coherence (or its absence).

IV. COUPLED LUTTINGER LIQUIDS - EXACT RESULTS

In this section, we present exact analytic results for the interliquid hopping rate using Luttinger liquid spectral functions. We are interested in the problem of N coupled Luttinger liquids, $N \to \infty$. At $O(t_\perp^2)$, however, our results are equivalent to those for $N = 2$, and we therefore consider the problem of two Luttinger liquids coupled by a spatially uniform, single particle hopping. The Hamiltonian is

$$H = H_{\rm LL}^{(1)} + H_{\rm LL}^{(2)} + t_\perp \sum_x \{c_\sigma^{(1)\dagger}(x) c_\sigma^{(2)}(x) + \text{h.c.}\} \tag{80}$$

As discussed in the previous section, the connection to the TLS-type physics is made by first bosonizing the Luttinger liquids, which then play the role of two baths of spin and charge bosons. Under the bosonization the interliquid hopping operators become exponentials of spin and charge boson creation and annihilation operators. They resemble the operators $e^{\pm i\Omega}$ of the TLS Hamiltonian, $H'_{\rm TLS}$. Further, the $t_\perp$ operator acts to raise the particle number of one chain by 1, and lower the other by 1, analogous to the action of the spin flip operators in the TLS. Moreover, the $t_\perp$ operator has a power law two-point function. Thus, it is very similar to the tunneling operator ($\sigma^+ e^{-i\Omega} + \sigma^- e^{i\Omega}$) in the ohmic regime of the TLS, and $H_{\rm LL}^{(1)} + H_{\rm LL}^{(2)}$ plays a role similar to the oscillator bath in the TLS.

Despite the striking similarity, however, there is no precise mapping of H to $H_{\rm TLS}$ for the simple reason that in the TLS problem there is just a *single* tunneling particle, whose tunneling is associated with a single exponential in boson creation and annihilation operators.

For fermions, the hopping can occur anywhere along the chains and so the operator which changes the relative particle number of two chains is coupled to an integral over space of exponentials in boson creation and annihilation operators. In a TLS the tunneling particle is *distinct* from the oscillator bath, while in the coupled LL problem there are N particles which can hop from liquid to liquid and, moreover, these particles are themselves the *source* of the dissipative bath.

Nevertheless, as discussed in the previous section, it is clear that $\delta N \equiv N_2 - N_1$, the particle number difference between the two liquids, is the most natural variable analogous to σ_z in the TLS, apart from the obvious difference that δN is not two-valued. In our previous work, we introduced the function $P(t)$ (not to be confused with the $P(t)$ in the TLS problem) defined by

$$P(t) \equiv |\langle O_1 O_2 | e^{iH_0 t} e^{-iHt} | O_1 O_2 \rangle|^2 \tag{81}$$

Here $|O_1 O_2\rangle$ denotes the product of the ground states $|O_1\rangle$, $|O_2\rangle$ of each Luttinger liquid in the absence of $t_\perp$. For $t < 0$, $H = H_{\rm LL}^{(1)} + H_{\rm LL}^{(2)}$, and at $t = 0$ the interliquid hopping is turned on. The particle number difference $\Delta N \equiv \delta N(t = 0)$ entails a Fermi momentum difference Δk and a chemical potential difference $\Delta\mu$. At $O(t_\perp^2)$, it is of no consequence, from the computational point of view, whether one calculates $P(t)$ or $\delta N(t)$, and we should emphasize that in all our work, reference to $P(t)$ is always to be taken to be at the $O(t_\perp^2)$ level. If one is interested in attempting calculations (in particular, numerical calculations, *e.g.* [54]) beyond lowest order in $t_\perp$, it is clear that $\delta N(t)$ is a more appropriate function to calculate than $P(t)$. The latter has a far too stringent condition on the need for the system to return to the initial state: at $O(t_\perp^2)$ this restriction is innocuous, but it clearly will not be at higher order [55].

The prescription, then, is to take the ground state $|0\rangle \equiv |O_1 O_2\rangle$ in the absence of interliquid hopping and to propagate that ground state for a time t. The nature of the departure of the propagated state from the initial state will determine the nature of the interliquid hopping processes.

Again, let us begin by supposing that, instead of being Luttinger liquids, the 1D liquids were free Fermi gasses. Then the Hamiltonian becomes a direct product $H = \otimes_k H_k$, where

$$H = \begin{pmatrix} E_k & t_\perp \\ t_\perp & E_k \end{pmatrix}$$

so that

$$P(t) = \cos^{2\Delta N}(t_\perp t)$$

Perturbation theory picks up the $O(t_\perp^2)$ term correctly,

$$P(t) \sim 1 - \Delta N t_\perp^2 t^2$$

This is precisely the type of behavior for which Golden Rule or extended Golden Rule, *i.e.* incoherent, type methods fail and for a very clear reason: quantum coherence is established separately for each k.

When interactions between electrons within a given liquid are included, H can no longer be written in this direct product form. We might suspect, however, that for true (Landau) Fermi liquids, where the Landau quasiparticle concept is valid, an approximate decomposition into a direct product of quasiparticle Hamiltonians would be possible. On the other hand, the situation for coupled Luttinger liquids, where the quasiparticle concept completely breaks down, is not at all obvious.

The calculation of $P(t)$ using space-time Green's functions was presented in our earlier work [6]. Here we shall use spectral function methods. This is more illuminating than the space-time Green's function method in that the "shape" of the spectral function which determines $P(t)$ provides a qualitative idea of the interliquid hopping processes. Moreover, the spectral function method allows us to more easily calculate key correlation functions, and hence $P(t)$, *exactly*.

To $O(t_\perp^2)$ we have

$$1-P(t)=2t_\perp^2 L\mathrm{Re}\int_0^t dt_1\int_0^{t_1}dt_2\int dx\left\{\langle c^{(1)}(x,t_1)c^{(1)\dagger}(0,t_2)\rangle\langle c^{(2)\dagger}(x,t_1)c^{(2)}(0,t_2)\rangle+(1\leftrightarrow 2)\right\} \tag{82}$$

where the superscripts on the electron operators label the chain in which the operator acts. The *interliquid hopping rate* $\Gamma(t)\equiv -dP(t)/dt$ can be written in a spectral function form as

$$\Gamma(t)=2t_\perp^2 L\int\frac{d\omega}{2\pi}\frac{\sin\omega t}{\omega}\{A_{12}(\omega)+A_{21}(\omega)\} \tag{83}$$

where

$$A_{ij}(\omega)=\int\frac{d\omega'}{2\pi}\int\frac{dk}{2\pi}\mathcal{J}_1^{(i)}(k,\omega')\mathcal{J}_2^{(j)}(k,\omega'-\omega) \tag{84}$$

and $\mathcal{J}_{1,2}(k,\omega)$ are the Fourier transforms of

$$\mathcal{J}_1(k,t)\equiv\langle c(k,t)c^\dagger(k,0)\rangle$$
$$\mathcal{J}_2(k,t)\equiv\langle c^\dagger(k,0)c(k,t)\rangle$$

In this paper we shall only consider the zero temperature limit, in which case

$$\mathcal{J}_{1,2}(k,\omega')=\theta_\pm(\omega'-\mu)\rho(k,\omega'-\mu) \tag{85}$$

where $\rho(k,\omega)$ is the electron spectral function as conventionally defined.

Physically, $A_{12}(\omega)$ is the effective spectral function governing hops in which an electron hops *to* liquid 1, *from* liquid 2, and $A_{21}(\omega)$ the opposite. There is then also a natural definition of interliquid hopping rates $\Gamma_{12}(t)$ and $\Gamma_{21}(t)$ satisfying $\Gamma_{12}(t)+\Gamma_{21}(t)=\Gamma(t)$.

At this point it should be emphasized that while, for concreteness, we restrict our discussion to the well-defined case of coupled 1D Luttinger liquids, the derivation of (83) is also quite generally applicable to the problem of coupled 2D liquids. All that one needs to know are the effective interliquid hopping spectral functions $A_{12}(\omega)$ and $A_{21}(\omega)$. The definition of these given in (84) may be easily extended to 2D by replacing k by $\mathbf{k}$ in the k-integrals and in the definition of $\mathcal{J}_1$ and $\mathcal{J}_2$.

Before presenting the calculation of $P(t)$ for coupled spin-1/2 Luttinger liquids, let us first observe how the coherence of interliquid hopping manifests itself in the case of coupled (Landau) Fermi liquids.

A. Free Fermi Gasses, and Fermi Liquids

To begin with, we note that for free Fermi gasses, $A_{12}(\omega) \propto \Delta\mu\delta(\omega)$ and $A_{21}(\omega) = 0$. Thus $\Gamma(t) \propto \Delta\mu\ t$, a clear signal of coherent hopping and hence of a fundamental rearrangement of the ground state.

In a true Fermi liquid the (retarded) Green's function is

$$G_R^{-1}(k,\omega) = Z^{-1}(\omega - E_k) + i\gamma\omega^2$$

where $Z \sim (1+2\gamma\Lambda/\pi)^{-1}$ is the quasiparticle renormalization factor, γ a (positive) parameter characterizing the strength of the electron-electron interactions, and Λ is an ultraviolet cutoff for these interactions. The spectral function is then given by $\rho(k,\omega) = -2\ \mathrm{Im}G_R(k,\omega)$, *i.e.*

$$\rho(k,\omega) = \frac{2Z^2\gamma\omega^2}{(\omega - E_k)^2 + Z^2\gamma^2\omega^4}$$

from which we obtain (see the Appendix for details)

$$A_{12}(\omega) \sim \frac{1}{v_F}\{Z^2\Delta\mu\delta(\omega) + \frac{1}{3\pi}Z^3\gamma\omega\}\,\theta_+(\omega + \Delta\mu) \tag{86}$$

$$A_{21}(\omega) \sim \frac{1}{3\pi v_F}\,Z^3\gamma\,\frac{(\omega - \Delta\mu)^3}{\omega^2}\,\theta_+(\omega - \Delta\mu) \tag{87}$$

Note that $Z \to 1$ if and only if $\gamma \to 0$, recovering the free Fermi gas result. From the point of view of coherence, only $\Gamma_{12}(t)$ is of interest. We see that $\Gamma_{12}(t)$ is a sum of a term $\propto Z^2\Delta\mu t$ representing fundamentally coherent processes, and a term $\propto \gamma Z^3 t^{-1}$ which is on the border of incoherent and irrelevant (*i.e.*, it is marginal). By choosing a sufficiently small $t_\perp$ one can find a time t such that $(1 - P(t))/N \ll 1$ (*i.e.* we are not outside of the reasonable range of our $O(t_\perp^2)$ expansion), yet the ratio of the coherent contribution to the incoherent contribution is arbitrarily large. This is true *regardless of how small Z is*. Thus, a perturbative calculation in $t_\perp$ does not reveal any likelihood of a loss of coherence of interliquid tunneling, and there is therefore no impediment to the formation of an interliquid band of width $\sim Zt_\perp$. This result is entirely consistent with what we would expect from a calculation based upon (Landau) quasiparticles, and is but another illustration of the power of the quasiparticle concept.

B. Luttinger Liquids

We now turn to the problem proper, that of Luttinger liquids coupled by interliquid, single-particle hopping. In order to calculate $A_{12}(\omega)$ and $A_{21}(\omega)$ we need the spectral functions $\mathcal{J}_1(\omega)$ and $\mathcal{J}_2(\omega)$ for a Luttinger liquid.

In [6] we used the space-time Luttinger liquid Green's functions, $G(x,t)$, to calculate $P(t)$, since these are more directly calculated within the bosonization framework than are the corresponding $G(k,\omega)$. The electron spectral function $\rho(k,\omega)$ *can* be calculated by direct Fourier transform of $G(x,t)$ [44], but it can actually be determined in a much simpler way which we shall now describe. The idea is to write the electron space-time Green's function

as a product of "fracton" Green's functions, where the fracton operators are exponentials of spin and charge boson operators. In contrast to the electron spectral function, the fracton spectral functions are sharp δ-functions. By "fusing" the fracton spectral functions together via convolution, one obtains simple integral expressions for the electron spectral function. From our point of view, the great utility of this method over the space-time approach is that it provides a method of explicitly obtaining the effective interliquid hopping spectral functions $A_{12}(\omega)$ and $A_{21}(\omega)$. These spectral functions have direct physical significance and can be used to obtain exact expressions for $P(t)$.

The general asymptotic form of the space-time electronic Green's function in a Luttinger liquid is

$$\langle \psi_{R,\uparrow}(x,t)\psi^{\dagger}_{R,\uparrow}(0,0)\rangle = \frac{e^{i(k_F x-\mu t)}}{2\pi a}\frac{(i\pi a/v_c\beta)^{\alpha+1/2}}{\left(\sinh\left[\frac{\pi(x+ia-v_c t)}{v_c\beta}\right]\right)^{\alpha+1/2}}\frac{(i\pi a/v_c\beta)^{\alpha}}{\left(\sinh\left[\frac{\pi(x+ia+v_c t)}{v_c\beta}\right]\right)^{\alpha}}\frac{(i\pi a/v_s\beta)^{1/2}}{\left(\sinh\left[\frac{\pi(x+ia-v_s t)}{v_s\beta}\right]\right)^{1/2}} \tag{88}$$

where β is inverse temperature, α is the so-called "anomalous" (or Luttinger liquid) exponent, v_c, v_s are the charge- and spin-velocities, and a is a short distance cutoff. Recall that 2α is the exponent which characterizes the singularity in $n(k)$ near k_F (see (5)) and is not to be confused with the α used in the discussion of the TLS problem.

The expression (88) is for right moving electrons (hence the subscript 'R'); but this is trivially changed to produce the left-moving part and hence the full Green' function. For our purposes, however, we need only consider the right-moving part.

For non-interacting electrons, $v_s = v_c = v_F$ and $\alpha = 0$. In this case we recover the free fermion propagator from (88). In the Hubbard model, $\alpha \sim U^2$, $v_c - v_s \sim U$ for small U, and $0 < \alpha < 1/16$ for all U. Larger exponents can occur, for example in extended Hubbard models.

The key observation to make at this point is that the Green's function is a product of fracton correlation functions,

$$\langle \psi_{R,\uparrow}(x,t)\psi^{\dagger}_{R,\uparrow}(0,0)\rangle = \frac{e^{i(k_F x-\mu t)}}{2\pi a}\left[G^{(\alpha+1/2)}_{c+}(x,t)\right]\left[G^{(\alpha)}_{c-}(x,t)\right]\left[G^{(1/2)}_{s}(x,t)\right] \tag{89}$$

We shall call $G^{(p)}(x,t)$ the Green's function of a fracton of *weight* p (further discussion may be found in the Appendix). The spectral function associated with $G^{(p)}(x,t)$ is therefore

$$\mathcal{J}^{(p)}_1(k,\omega) = \left(\frac{i\pi a}{v\beta}\right)^p 2\pi\delta(\omega - vk)\int_{-\infty}^{\infty}\frac{dz\; e^{-ikz}}{\left[\sinh\left(\frac{\pi(z+ia)}{v\beta}\right)\right]^p} \tag{90}$$

where v is the velocity of the relevant charge or spin bosonic excitation. In the zero temperature limit this simplifies to

$$\mathcal{J}^{(p)}_1(k,\omega) \stackrel{\beta\to\infty}{=} \frac{4\pi^2}{\Gamma(p)}a^p\; k^{p-1}\theta_+(\omega)\delta(\omega - vk) \tag{91}$$

It is then a simple matter to obtain the *electron* spectral function by convolving the appropriate fracton spectral functions as determined by (89). We find

$$\mathcal{J}_1(k,\omega)\propto \int dk_1 dk_2 dk_3 \int d\omega_1 d\omega_2 d\omega_3 \delta(\omega-\mu-\sum_i \omega_i)\delta(k-k_F-\sum_i k_i)$$
$$\delta(\omega_1-v_c k_1)\theta_+(\omega_1)(\omega_1/v_c)^{\alpha-1/2}\delta(\omega_2-v_s k_2)\theta_+(\omega_2)(\omega_2/v_s)^{-1/2}\delta(\omega_3+v_c k_3)\theta_+(\omega_3)(\omega_3/v_c)^{\alpha-1}$$
$$\propto \int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega-\mu-\sum_i \omega_i)\delta\left(k-k_F-\frac{(\omega_1-\omega_3)}{v_c}-\frac{\omega_2}{v_s}\right)$$
$$(\omega_1/v_c)^{\alpha-1/2}(\omega_2/v_s)^{-1/2}(\omega_3/v_c)^{\alpha-1} \tag{92}$$

The various singularities near $\omega = \pm v_c k,\ v_s k$ can be readily determined from this expression and are illustrated in figure 4.

We again emphasize that the fracton spectral functions are sharp, due to the fact that a fracton is an exponential of one of the free bosonic fields which have well defined energy-momentum dispersion relations. It is therefore quite natural in this language to think of the electron as a *composite* particle made up of *spin- and charge-fracton quasiparticles*.

For reasons of pedagogy, it is convenient to first consider the case of *chiral* Luttinger liquid models which are forward-scattering-only (FSO), *i.e.* there is no coupling between left- and right-moving electrons. Then we shall consider the generic Luttinger liquid case.

1. Chiral Luttinger liquid

This model exhibits spin-charge separation, but no anomalous exponent [56]. The eigenexcitations are spin and charge bosons with velocities v_s and v_c, respectively, and $v_c - v_s \equiv \Delta v > 0$. The electron Green's function may be obtained from (88) by simply setting $\alpha \to 0$. There is no "holonic backflow", and so the electron spectral function may be written as the convolution of the spin fracton spectral function and a single charge fracton spectral function

$$\mathcal{J}_{1,2}(k,\omega) = \frac{1}{(2\pi)^2}\int dk_1 \int dk_2 \int d\omega_1 \int d\omega_2 \mathcal{J}^{(c)}_{1,2}(k_1,\omega_1)\mathcal{J}^{(s)}_{1,2}(k_2,\omega_2)$$
$$\delta(\omega-\mu-\omega_1-\omega_2)\delta(k-k_F-k_1-k_2) \tag{93}$$

where

$$\mathcal{J}^{(c)}_1(k,\omega) = 4\pi^{3/2}a^{1/2}\theta_+(\omega)\delta(\omega-v_c k)\left(\frac{\omega}{v_c}\right)^{-1/2} \tag{94}$$

$$\mathcal{J}^{(s)}_1(k,\omega) = 4\pi^{3/2}a^{1/2}\theta_+(\omega)\delta(\omega-v_s k)\left(\frac{\omega}{v_s}\right)^{-1/2} \tag{95}$$

$$\mathcal{J}^{(c)}_2(k,\omega) = 4\pi^{3/2}a^{1/2}\theta_-(\omega)\delta(\omega-v_c k)\left(\frac{-\omega}{v_c}\right)^{-1/2} \tag{96}$$

$$\mathcal{J}^{(s)}_2(k,\omega) = 4\pi^{3/2}a^{1/2}\theta_-(\omega)\delta(\omega-v_s k)\left(\frac{-\omega}{v_s}\right)^{-1/2} \tag{97}$$

We have changed notation slightly, the superscripts 'c' and 's' referring to 'charge' and 'spin'. Note that, as in the case of a free Fermi gas, there is no response to $t_\perp$ if $\Delta\mu = 0$, i.e. both $A_{12}(\omega)$ and $A_{12}(\omega)$ vanish. This is the result of there being a step-function singularity

at the Fermi surface, introducing the well-known problem of the inability to incorporate "anomalous diagrams" in zero temperature perturbation theory [53].

For $\Delta\mu > 0$ we find (see Appendix) the rather simple expression

$$\begin{aligned} A_{12}(\omega) &= \frac{1}{\Delta v}\theta_+((v_c - v)\Delta k - \omega)\theta_+(\omega - (v_s - v)\Delta k) \\ A_{21}(\omega) &= 0 \end{aligned} \tag{98}$$

where we have introduced the velocity v corresponding to the chemical potential shift, $\Delta\mu \equiv v\Delta k$. That is, $A_{12}(\omega)$ has constant non-vanishing weight in the interval $\omega \in [(v_s - v)\Delta k, (v_c - v)\Delta k]$. Note that, in the limit $\Delta v \to 0$, this step function of width Δv, height $\propto 1/\Delta v$, goes over to a δ-function, as it should, for the limit $\Delta v \to 0$ is the limit of free electrons. For $\Delta v \neq 0$, $A_{12}(\omega)$ is peaked around $\omega = 0$, but has a width $\tau_{\Delta k}^{-1} \sim \Delta k \Delta v$.

It is then a simple matter to calculate $\Gamma(t)$ for a chiral Luttinger liquid. We have, using (83),

$$\begin{aligned} \Gamma(t) &= \frac{1}{\pi\Delta v} t_\perp^2 L \int_{(v_s - v)\Delta k}^{(v_c - v)\Delta k} d\omega \frac{\sin \omega t}{\omega} \\ &= \frac{1}{\pi\Delta v} t_\perp^2 L \left\{ \mathrm{Si}((v_c - v)\Delta k t) - \mathrm{Si}((v_s - v)\Delta k t) \right\} \end{aligned} \tag{99}$$

where $\mathrm{Si}(x)$ is the Sine-integral function. The behavior of $\mathrm{Si}(x)$ for small $|x|$ is $\mathrm{Si}(x) \sim x$ so that provided $(v_c - v)\Delta k t \ll 1$ and $(v - v_s)\Delta k t \ll 1$, we have

$$\begin{aligned} \Gamma(t) &\sim \frac{1}{\pi\Delta v} t_\perp^2 L \Delta v \Delta k t \\ &= \Delta N t_\perp^2 t \end{aligned} \tag{100}$$

Thus, for all times $t \lesssim \tau_{\Delta k}$, $\Gamma(t)$ behaves just as it does for coupled free Fermi gasses. Moreover, $\tau_{\Delta k}$ can be made arbitrarily long by choice of sufficiently small Δk, while remaining in the perturbative regime, $N^{-1}\int_0^t \Gamma(t')dt' \ll 1$. The physics here is that $\tau_{\Delta k}$ plays the role of an intrinsic lifetime due to spin-charge separation, but this lifetime diverges as k_F is approached (*i.e.* as $\Delta k \to 0$). This is the only lifetime in the model.

We conclude, therefore, that for coupled chiral Luttinger liquids, as for coupled Fermi liquids, there is no obvious impediment to interliquid coherence for arbitrarily small $t_\perp$.

2. Spinless Luttinger liquid

After the chiral case, the next simplest Luttinger liquid to consider is that formed by interacting spinless fermions. The space-time Green's function has the form (88) with $v_c = v_s$ (*i.e.* $\Delta v = 0$). In place of (92) we have

$$\mathcal{J}_1(k,\omega) \propto \int_0^\infty d\omega_1 d\omega_2 \delta(\omega - \mu - \omega_1 - \omega_2)\delta\left(k - k_F - \frac{(\omega_1 - \omega_2)}{v_c}\right)(\omega_1/v_c)^\alpha(\omega_2/v_c)^{\alpha-1} \tag{101}$$

for the spinless fermion spectral function. The hole spectral function is

$$\mathcal{J}_2(k,\omega) \propto \int_0^\infty d\omega_1' d\omega_2' \delta(\omega - \mu + \omega_1' + \omega_2') \delta\left(k - k_F + \frac{(\omega_1' - \omega_2')}{v_c}\right) (\omega_1'/v_c)^\alpha (\omega_2'/v_c)^{\alpha-1} \quad (102)$$

The effective interliquid hopping spectral functions $A_{12}(\omega)$ and $A_{21}(\omega)$ are then obtained from (84). We find

$$A_{12}(\omega) = \frac{1}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c} \theta_+(\omega - (v_c - v)\Delta k)(\omega + (v_c + v)\Delta k)^{2\alpha+1}(\omega - (v_c - v)\Delta k)^{2\alpha-1} \quad (103$$

$$A_{21}(\omega) = \frac{1}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c} \theta_+(\omega - (v_c + v)\Delta k)(\omega - (v_c + v)\Delta k)^{2\alpha+1}(\omega + (v_c - v)\Delta k)^{2\alpha-1} \quad (104$$

In the noninteracting case ($\alpha = 0$ and $v_c = v$) $A_{21}(\omega)$ vanishes and so interliquid hopping occurs only from liquid 2 to liquid 1. It is straightforward to see that for $\alpha \neq 0$ we need only examine $A_{12}(\omega)$, and hence $\Gamma_{12}(t)$, to address the coherence issue. Representative plots of $A_{12}(\omega)$ for various α are shown in Fig. 5.

In calculating $\Gamma_{12}(t)$ it is simplest first of all to consider its time derivative, since this is just a Fourier transform of the spectral function $A_{12}(\omega)$, and therefore easier to deal with analytically:

$$\frac{d\Gamma_{12}(t)}{dt} = \frac{t_\perp^2 L}{\pi} \int_{-\infty}^{\infty} d\omega \ \cos\omega t \ A_{12}(\omega) \quad (105)$$

Inserting the above expression for $A_{12}(\omega)$ gives (omitting prefactors)

$$\frac{d\Gamma_{12}(t)}{dt} \propto t^{-(1+4\alpha)} \mathrm{Re}\left\{ e^{i(v_c - v)\Delta k t} \int_0^\infty d\theta \ \theta^{2\alpha-1}(\theta + x)^{2\alpha+1} e^{i\theta} \right\} \quad (106)$$

where, for convenience, we have introduced the variable $x = 2v_c \Delta k t$.

The integral may be done via contour integration, and the final result is

$$\begin{aligned} \frac{d\Gamma_{12}(t)}{dt} = \frac{t_\perp^2 L}{\pi} \frac{1}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c} \frac{1}{\Gamma(1-2\alpha)} t^{-(1+4\alpha)} \\ \mathrm{Re}\{ e^{i(v_c - v)\Delta k t} [i e^{i2\pi\alpha} \Gamma(1-2\alpha)\Gamma(1+4\alpha) \ {}_1F_1(-1-2\alpha, -4\alpha; -ix) \\ + \frac{1}{2} \frac{(1+2\alpha)}{(1+4\alpha)} \Gamma(2\alpha)\Gamma(1-4\alpha) \ x^{1+4\alpha} \ {}_1F_1(2\alpha, 2+4\alpha; -ix)] \} \end{aligned} \quad (107)$$

where ${}_1F_1$ is the confluent hypergeometric function.

Equation (107) is an *exact* result for the (time derivative of the) interliquid hopping rate, to lowest order in $t_\perp$, obtained by the use of fracton spectral functions. It allows us to extract the important physics. To begin with, even when the particle number difference ΔN vanishes (*i.e.* $\Delta k = 0$) there are *incoherent* interliquid transitions, for

$$\frac{d\Gamma_{12}^{\Delta k=0}(t)}{dt} = -\frac{t_\perp^2 L}{\pi} \frac{\Gamma(1+4\alpha)}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \sin(2\pi\alpha) \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c} t^{-(1+4\alpha)} \quad (108)$$

thus

$$\Gamma_{12}^{\Delta k=0}(t) = \frac{t_\perp^2 L}{\pi} \frac{\Gamma(1+4\alpha)}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \frac{\sin(2\pi\alpha)}{4\alpha} \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c} t^{-4\alpha} \quad (109)$$

Such relevant, incoherent interliquid hopping does not occur for coupled Fermi liquids, nor for chiral Luttinger liquids. In a Fermi liquid, coherent interliquid hopping manifests itself within perturbation theory via a $t_\perp^2 \Delta N\, t$ contribution to $\Gamma_{12}(t)$, so we need to examine the Δk dependence of (107).

Upon using the expansion

$$ {}_1F_1(a,b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \ldots $$

we deduce from (107) that as $\alpha \to 0$ only the $O(x)$ and $O(x^{1+4\alpha})$ terms survive: all other terms are proportional to α. Indeed, one can readily check that the contribution of these two terms correctly recovers the free fermion result in the limit $\alpha \to 0$ (and $v_c - v \to 0$). Moreover, it makes little physical sense to suppose that terms of $O(x^2)$ or higher (*i.e.* terms of $O(n^2)$ or higher, where $n = \Delta N/L$ is the density difference between the liquids) are important in determining the coherence or incoherence of *single particle* hopping. We therefore retain only the $O(x^0)$, $O(x)$ and $O(x^{1+4\alpha})$ terms. Again, the latter two continuously develop into the free fermion limit of coherent interliquid single particle hopping as $\alpha \to 0$.

Retaining only these terms gives

$$ \begin{aligned} \frac{d\Gamma_{12}(t)}{dt} = \frac{t_\perp^2 L}{\pi} \frac{1}{\Gamma(2\alpha)\Gamma(2+2\alpha)} \left(\frac{a}{2v_c}\right)^{4\alpha} \frac{1}{2v_c}\, \cos[(v_c - v)\Delta k t]\, t^{-(1+4\alpha)} \\ \{(1+2\alpha)\left\{ -\sin(2\pi\alpha)\frac{\Gamma(1+4\alpha)}{(1+2\alpha)} + \cos(2\pi\alpha)\Gamma(4\alpha)\, x + \frac{\Gamma(2\alpha)\Gamma(1-4\alpha)}{2(1+4\alpha)\Gamma(1-2\alpha)}\, x^{1+4\alpha} \right\} \\ - \tan[(v_c - v)\Delta k t]\, \cos(2\pi\alpha)\Gamma(1+4\alpha)\} \end{aligned} \tag{110} $$

For $2\pi\alpha \ll 1$ we have

$$ \frac{d\Gamma_{12}(t)}{dt} \approx \frac{t_\perp^2 L}{\pi}\frac{1}{2v_c}\, \cos[(v_c - v)\Delta k t]\, t^{-(1+4\alpha)} \left\{ -4\pi\alpha^2 + v_c \Delta k\, t + \frac{1}{2}(2v_c \Delta k\, t)^{1+4\alpha} \right\} \tag{111} $$

Let us now discuss the physics of each of the terms in (110). As already remarked, the Δk-independent term represents fundamentally incoherent interliquid hops. By analogy with the situation in the TLS [38], if the term linear in Δk decays slower than t^{-1} it should be interpreted as a potentially coherent term. For $\alpha > 1/2$, the term linear in Δk is irrelevant in the sense that its contribution to $P(t)$ is long-time convergent. This is consistent with the renormalization group characterization of the relevance/irrelevance of the interliquid hopping operator.

The $O(\Delta k^{1+4\alpha})$ term requires careful thought to interpret when $\alpha \neq 0$. The key thing to note is that the oscillatory prefactor $\cos[(v_c - v)\Delta k t]$ will force $\Gamma_{12}(t)$ to be essentially time-independent for times $t \gtrsim [(v_c - v)\Delta k]^{-1}$. This prefactor is analogous to the prefactor which occurs in the tunneling rate of a TLS when the two levels are not energy degenerate. It indicates that, unlike for hopping between Fermi liquids, hopping between Luttinger liquids is never an energy degenerate process. This is another important factor working against coherent interliquid hopping.

We now utilize this oscillatory prefactor to interpret the $O(\Delta k^{1+4\alpha})$ term. In order for this term to be of coherent type, one must remain at times short enough to avoid

the cutoff effect of the $\cos[(v_c - v)\Delta k t]$ prefactor. The maximum possible Δk for a given time t is $\Delta k^{\max} \sim [(v_c - v)t]^{-1}$. The $O(\Delta k^{1+4\alpha})$ term in $d\Gamma/dt$ is therefore bounded by $\sim \Delta k\, t^{-4\alpha}/(v_c - v)^{4\alpha}$ which has the same form as the term linear in Δk. It therefore suffices to consider only the latter term.

We now observe that for $\alpha > 1/4$ the $O(\Delta k)$ term in $d\Gamma/dt$ behaves as $t^{-4\alpha}$, which is weaker than t^{-1}. In fact, for $\alpha = 1/4$ this term is identically zero, reflecting the fact that at this particular value of α, the linear-in-Δk contribution to $\Gamma(t)$ is actually time-independent. Therefore, for $1/4 \leq \alpha < 1/2$ [38] the $O(t_\perp^2)$ calculation indicates that interliquid single particle hopping is completely incoherent, *despite the relevance of $t_\perp$ in the RG sense*. This is completely analogous to the regime in the TLS model where inter-state tunneling is relevant but non-oscillatory. Indeed, ignoring the $\cos[(v_c - v)\Delta k t]$ prefactor for the moment, if the term linear in Δk was the *only* contribution to Γ_{12} the result would be formally equivalent, at $O(t_\perp^2)$, to that of ΔN TLS's in parallel. The modification from the FL result $\Gamma(t) \sim \Delta N\, t_\perp^2 t$ to a behavior $\Gamma(t) \sim \Delta N\, t_\perp^2 \Lambda^{4\alpha} t^{1-4\alpha}$ ($\Lambda \sim a/v_c$ being an UV cutoff) would be completely analogous to the behavior of a TLS upon turning on coupling to the ohmic bath. In particular, in the nomenclature of the TLS problem the interliquid transport of electrons would be localized when $\alpha > 1/2$, purely incoherent when $1/4 < \alpha < 1/2$, and coherent if $0 \leq \alpha < 1/4$. At the very least then, there would be a nontrivial incoherent interliquid hopping phase for $1/4 < \alpha < 1/2$. However, unlike in the (degenerate) TLS problem, for the problem of coupled Luttinger liquids there are additional factors enhancing incoherence over and above the time exponent of the $O(\Delta k)$ term. The question then arises as to the nature of interliquid transport in the regime $0 < \alpha < 1/4$.

The factors enhancing incoherence are as follows. First of all, there are the incoherent processes contributing to the Δk-independent term. This term represents a "channel" in which the electrons hop in a purely incoherent way from liquid to liquid.

Secondly, as already remarked upon, there is a "dephasing" prefactor $\cos[(v_c - v)\Delta k t]$ which is analogous to a biassing term in a TLS. This will quite generally enhance incoherence.

Thus, for $0 < \alpha < 1/4$ the interliquid hopping rate is essentially a sum of incoherent and coherent parts, $\Gamma_{12}(t) = \Gamma_{12}^{\rm incoh}(t) + \Gamma_{12}^{\rm coh}(t)$. Alternatively, one can consider the integrated transition probabilities up to time t and write $P_{12}(t) = P_{12}^{\rm incoh}(t) + P_{12}^{\rm coh}(t)$. The key point to recognize is that the coherent term remains so only for times $t \lesssim [(v_c - v)\Delta k]^{-1}$. As such, $P_{12}^{\rm coh}(t)$ is bounded above in magnitude by $\sim t_\perp^2 v_c \Lambda^{4\alpha} t^{1-4\alpha}/(v_c - v)$ so that, approximately,

$$\frac{P_{12}^{\rm incoh}(t)}{P_{12}^{\rm coh}(t)} \gtrsim \alpha \frac{(v_c - v)}{v_c}$$

which is *independent* of $t_\perp$. In particular, the purely incoherent channel cannot eliminated in the $t_\perp \to 0$ limit, as it would have been were the dephasing factor absent. Thus, as we decrease α from $\alpha = 1/4$ we expect incoherence to be stabilized down to some critical value $\alpha_c < 1/4$ by a combination of the purely incoherent term, which will always dominate if Δk is too small, and the dephasing prefactor of the TLS-like coherent term linear in Δk which kills coherence of this term if Δk is too large. The reader should contrast this with the situation in a chiral Luttinger liquid discussed earlier. In that case, the absence of a purely incoherent interliquid hopping term allows Δk to be chosen sufficiently small for a given $t_\perp$ that dephasing factors can be suppressed to an arbitrary degree.

Finally, there is a further effect which will push α_c even lower, namely the influence of interliquid hops upon one another via intraliquid interactions. Interhop correlations (effectively absent for hopping of electrons between Fermi liquids) are not included in our $O(t_\perp^2)$ calculation. It is physically clear that such correlations will only hinder coherence and an estimate of their effect has already been given in the previous section. In particular, if the effect of incoherent hops in disrupting potentially coherent hops is such that the the amplitude for the latter is premultiplied by any function of time asymptoting to zero for $t \gg (t_\perp^R)^{-1}$, it can be shown that in the limit $t_\perp \to 0$ the incoherent phase extends over all $0 < \alpha < 1/2$. In that case, one would have $\alpha_c = 0$ and the operations of turning on in-liquid interactions and of turning on interliquid hopping would not commute.

3. Generic (Spinny) Luttinger liquid

We now turn to the generic case where there is spin-charge separation, $v_c - v_s = \Delta v > 0$, *and* an anomalous exponent, α. This case is relevant, for example, to the 1D Hubbard model, and to most physical models which are not "chiral".

For ease of notation we shall omit constants of proportionality in most expressions, restoring them only at the end. To begin with, we have the electron spectral functions

$$\mathcal{J}_1(k,\omega) \propto \int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega - \mu - \sum_i \omega_i)\delta\left(k - k_F - \frac{(\omega_1 - \omega_3)}{v_c} - \frac{\omega_2}{v_s}\right) \\ (\omega_1/v_c)^{\alpha-1/2}(\omega_2/v_s)^{-1/2}(\omega_3/v_c)^{\alpha-1}$$

and

$$\mathcal{J}_2(k,\omega) \propto \int_0^\infty d\omega_1' d\omega_2' d\omega_3' \delta(\omega - \mu + \sum_i \omega_i')\delta\left(k - k_F + \frac{(\omega_1' - \omega_3')}{v_c} + \frac{\omega_2'}{v_s}\right) \\ (\omega_1'/v_c)^{\alpha-1/2}(\omega_2'/v_s)^{-1/2}(\omega_3'/v_c)^{\alpha-1}$$

The effective spectral function for hopping is then given by (84)

$$A_{12}(\omega) \propto \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} \int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega' - \sum_i \omega_i)\delta\left(k - k_F - \frac{(\omega_1 - \omega_3)}{v_c} - \frac{\omega_2}{v_s}\right) \omega_1^{\alpha-1/2}\omega_2^{-1/2}\omega_3^{\alpha-1} \\ \int_0^\infty d\omega_1' d\omega_2' d\omega_3' \delta(\omega - \omega' - \Delta\mu + \sum_i \omega_i')\delta\left(k - k_F + \frac{(\omega_1' - \omega_3')}{v_c} + \frac{\omega_2'}{v_s}\right) (\omega_1')^{\alpha-1/2}(\omega_2')^{-1/2}(\omega_3')^{\alpha-1}$$

Some straightforward, if laborious, algebra (see Appendix) reduces this to

$$A_{12}(\omega) \propto I(a,b) \\ = \int_0^\infty dx \int_0^\infty dy x^{-1/2} y^{-1/2} (a - (x+y))^{2\alpha} (b + (x+y))^{2\alpha-1} \theta_+(a - (x+y))\theta_+(b + (x+y))$$

where

$$a = \frac{v_s}{\bar{v}}(\omega + (v + v_c)\Delta k) \\ b = \frac{v_s}{\Delta v}(\omega + (v - v_c)\Delta k)$$

and we have introduced $\bar{v} \equiv v_c + v_s$.

Further simplification of the double integral is possible, and we finally arrive at the following exact result, illustrated in Fig. 6

$$\begin{aligned} &= 0, \quad \omega < (v_s - v)\Delta k \\ A_{12}(\omega) &= \frac{1}{\Gamma(1+4\alpha)} \left(\frac{a^2}{\bar{v}\Delta v}\right)^{2\alpha} \left(\frac{1}{\Delta v}\right) (\omega + (v - v_s)\Delta k)^{4\alpha}, \quad (v_s - v)\Delta k < \omega < (v_c - v)\Delta k \\ &= \frac{1}{(1+2\alpha)} \frac{1}{\Gamma(2\alpha)\Gamma(1+2\alpha)} \frac{1}{\bar{v}} \left(\frac{a}{2v_c}\right)^{4\alpha} (\omega + (v_c + v)\Delta k)^{2\alpha+1} (\omega - (v_c - v)\Delta k)^{2\alpha-1} \\ &\quad {}_2F_1\left(1, 1-2\alpha; 2+2\alpha; -\left(\frac{\Delta v}{\bar{v}}\right)\left[\frac{\omega + (v_c + v)\Delta k}{\omega - (v_c - v)\Delta k}\right]\right), \quad \omega > (v_c - v)\Delta k \end{aligned} \tag{112}$$

For notational purposes, we shall divide $A_{12}(\omega)$ into its natural "high" and "low" frequency parts according to

$$A_{12}(\omega) = \theta_+(\omega - (v_s - v)\Delta k)\theta_+((v_c - v)\Delta k - \omega) A_{12}^{\text{low}}(\omega) + \theta_+(\omega - (v_c - v)\Delta k) A_{12}^{\text{high}}(\omega) \tag{113}$$

Note that this definition is based on the structure of A and not on the nature of the hopping and is not equivalent to the division into energies "high" and "low" compared to the renormalized hopping, $t_\perp^R$ which we consider elsewhere in this paper.

Note that both the chiral and spinless limits are correctly reproduced by this spectral function. The chiral limit is $\alpha \to 0$ in which case $A_{12}^{\text{high}}(\omega) \propto \Gamma^{-1}(2\alpha) \sim 2\alpha \to 0$, and $A_{12}^{\text{low}}(\omega) \to (\Delta v)^{-1}$. For the spinless limit, we take $\Delta v \to 0$, so ${}_2F_1(a, b, c, -z(\Delta v/\bar{v})) \to {}_2F_1(a, b, c, 1) = 1$ recovering the correct expression for $A_{12}^{\text{high}}(\omega)$, and we can set $A_{12}^{\text{low}}(\omega) \to 0$ because its integrated weight $\sim (\Delta v)^{2\alpha} \stackrel{\Delta v \to 0}{\to} 0$.

Some qualitative remarks are in order. The first is to note that $A_{12}(\omega)$ has non-zero weight over an energy range from just below $\omega = 0$ all the way up to the ultraviolet cutoff. As for the spinless case, $A_{12}(\omega) \propto \omega^{4\alpha}$ at high frequencies, $\omega \gg (v_c - v)\Delta k$. Again, for $\Delta k = 0$ (*i.e.* $\Delta N = 0$) this is all there is to $A_{12}(\omega)$. A spectral function $\propto \omega^{4\alpha}$ implies a $P(t) \propto t^{1-4\alpha}$ which describes manifestly incoherent interliquid hopping. We again emphasize that such incoherent processes are not present for coupled Fermi liquids, nor for chiral Luttinger liquids.

Upon comparing Figs. 5 and 6 we see that one feature of $A_{12}(\omega)$ differentiating the spinny from the spinless Luttinger liquid is the absence of any low frequency singularity. This follows by using the integral form for ${}_2F_1$:

$$\begin{aligned} {}_2F_1(1, 1-2\alpha; 2+2\alpha; -z) &= \frac{\Gamma(2+2\alpha)}{\Gamma(1+2\alpha)} \int_0^1 dt\ (1-t)^{2\alpha} (1+tz)^{2\alpha-1} \\ &\propto z^{2\alpha-1} \end{aligned} \tag{114}$$

in the limit $z \gg 1$. Thus in the vicinity of $\omega - (v_c - v)\Delta k = 0$, the contribution $\sim (\omega - (v_c - v)\Delta k)^{1-2\alpha}$ from the ${}_2F_1$ part of $A_{12}^{\text{high}}(\omega)$ cancels the $\sim (\omega - (v_c - v)\Delta k)^{2\alpha-1}$ piece, and there is no singularity.

Recall that Fermi's Golden Rule, which describes incoherent decay from an initially prepared state, cannot be applied if the effective spectral function of final states is singular, and/or has too narrow support. The nonsingular nature of $A_{12}(\omega)$ and the fact that it has support over the full energy range, gives us good reason to believe that incoherent methods like the Golden Rule will work. We reiterate that this was not the case for coupled Fermi liquid or chiral Luttinger liquids.

We now turn to evaluating the interliquid hopping rate. Again, it is easiest to consider $d\Gamma(t)/dt = -d^2P(t)/dt^2$, for then the integrals simplify and we obtain the following exact result:

$$\begin{aligned}
&\frac{d\Gamma_{12}(t)}{dt} = \frac{d\Gamma_{12}^{\rm low}(t)}{dt} + \frac{d\Gamma_{12}^{\rm high}(t)}{dt} \\
&\frac{d\Gamma_{12}^{\rm low}(t)}{dt} = \frac{t_\perp^2 L}{\pi}\frac{1}{\Gamma(1+4\alpha)}\left(\frac{a^2}{\bar{v}\Delta v}\right)^{2\alpha}\frac{1}{\Delta v}\,t^{-(1+4\alpha)}\,{\rm Re}\left\{e^{i(v_s-v)\Delta k t}\int_0^{\Delta v\Delta k t} d\theta\;\theta^{4\alpha}e^{i\theta}\right\} \qquad (115)\\
&\frac{d\Gamma_{12}^{\rm high}(t)}{dt} = \frac{t_\perp^2 L}{\pi}\frac{1}{(1+2\alpha)}\frac{1}{\Gamma(2\alpha)\Gamma(1+2\alpha)}\left(\frac{a}{2v_c}\right)^{4\alpha}\frac{1}{\bar{v}}\,t^{-(1+4\alpha)} \\
&{\rm Re}\left\{e^{i(v_c-v)\Delta k t}\int_0^\infty d\theta\;\theta^{2\alpha-1}(\theta+2v_c\Delta k t)^{2\alpha+1}e^{i\theta}\;{}_2F_1\left(1,1-2\alpha;2+2\alpha;\left(\frac{-\Delta v}{\bar{v}}\right)\left(\frac{\theta+2v_c\Delta k t}{\theta}\right)\right)\right\}
\end{aligned}$$

We will now show that, as expected on physical grounds, the effect of spin-charge separation is to enhance incoherence further beyond the effects discussed in the spinless case. We begin with $d\Gamma_{12}^{\rm high}/dt$ which requires evaluation of the integral

$$\int_0^\infty d\theta\;\theta^{2\alpha-1}(\theta+x)^{2\alpha+1}e^{i\theta}\;{}_2F_1\left(1,1-2\alpha;2+2\alpha;\left(\frac{-\Delta v}{\bar{v}}\right)\left(\frac{\theta+x}{\theta}\right)\right)$$

Unfortunately, we do not know of an exact result for this integral when $\Delta v \neq 0$. Moreover, its nonanalytic nature makes an asymptotic analysis nontrivial.

The important effect of spin-charge separation, as has already been pointed out, is that it removes the $\theta \to 0$ singularity in the integrand. For $\Delta v = 0$ the $O(x)$ contribution from this integral is proportional to α^{-1}. In contrast, in the almost chiral limit ($\Delta v \neq 0$, $\alpha \ll 1$) the integral can be approximated by setting $\alpha = 0$, *i.e.* evaluating

$$\int_0^\infty d\theta\;\theta^{-1}(\theta+x)e^{i\theta}\;{}_2F_1\left(1,1-2\alpha;2+2\alpha;\left(\frac{-\Delta v}{\bar{v}}\right)\left(\frac{\theta+x}{\theta}\right)\right)$$

This integral can be evaluated exactly using ${}_2F_1(1,1;2,-z) = \log(1+z)/z$. The result is

$$i\left(\frac{\bar{v}}{\Delta v}\right)\left\{\gamma + \log\left(\frac{-i\Delta v x}{\bar{v}}\right) + \exp\left(\frac{-i\Delta v x}{\bar{v}+\Delta v}\right)E_1\left(\frac{-i\Delta v x}{\bar{v}+\Delta v}\right)\right\}$$

where $\gamma \approx 0.577$ is Euler's constant, and E_1 is the exponential integral function. The small x behavior is

$$-\frac{\bar{v}\,x}{\bar{v}+\Delta v}\log\left(\frac{\Delta v\,x}{\bar{v}+\Delta v}\right)$$

(which is singular in the $\Delta v \to 0$ limit, as required, but otherwise finite).

In the spinless case, the incoherent (Δk-independent) contribution to $d\Gamma/dt$ is $O(\alpha)$ for small α, while the coherent $O(\Delta k)$ term is $O(\alpha^0)$. The free Fermi gas limit is correctly recovered as $\alpha \to 0$. In contrast, using the results above, for an almost chiral Luttinger liquid both the incoherent and coherent contributions to $d\Gamma^{\rm high}/dt$ are $O(\alpha)$. In fact, the dominant coherent contribution comes from the low frequency part of the spectral function. We have

$$\begin{aligned}\frac{d\Gamma_{12}^{\rm low}(t)}{dt} &\propto t^{-(1+4\alpha)}\,{\rm Re}\left\{e^{i(v_s-v)\Delta k t}\int_0^{\Delta v\Delta k t} d\theta\;\theta^{4\alpha}e^{i\theta}\right\}\\ &= t^{-(1+4\alpha)}\left\{\cos[(v_s-v)\Delta k\,t]\int_0^{\Delta v\Delta k t} d\theta\;\theta^{4\alpha}\cos\theta + O((\Delta k)^2)\right\}\end{aligned}$$

(note that this vanishes for $\Delta k = 0$). The integral is bounded above by $\sin(\Delta k\,\Delta v\,t)$ which provides a short time expansion. At long times ($\Delta k\,\Delta v\,t \gg 1$)

$$\int_0^{\Delta v\Delta k t} d\theta\;\theta^{4\alpha}\cos\theta \to -\sin(2\pi\alpha)\Gamma(1+4\alpha)$$

and the contribution crosses over to a purely incoherent time dependence.

The almost chiral limit provides an illustration of the enhancement of incoherence due to spin-charge separation. In this limit the coherent contribution to $d\Gamma/dt$ is dominated by the low frequency contribution and for small α is bounded by

$$\frac{t_\perp^2 L}{\pi}\frac{1}{\Delta v}\,t^{-(1+4\alpha)}\;\cos[(v_c-v)\Delta k t]\sin[(v_c-v_s)\Delta k t] \tag{116}$$

In comparison, the coherent contribution in the spinless case at small α is

$$\frac{t_\perp^2 L}{\pi}\Delta k\,t^{-4\alpha}\;\cos[(v_c-v)\Delta k t]$$

Equation (116) is equivalent to this at short times, $t \lesssim [(v_c-v_s)\Delta k)]^{-1}$ (using $v_c - v = v - v_s$ at small α), but at finite time the spin-charge separation provides a further decohering factor beyond those present in the spinless limit. As such, spin-charge separation can lead to a smaller α_c in the $t_\perp \to 0$ limit.

4. Summary

We have shown, in agreement with the arguments of the previous section, that a consideration of the exact expressions for the interliquid hopping rate presented above obtained via spectral function methods, leads to the conclusion that there is a *nontrivial incoherent phase* in the small $t_\perp$ limit for all $\alpha_c < \alpha < 1/2$. In general, $\alpha_c < 1/4$ and is a decreasing function of $\Delta v/\bar{v}$. Open questions are whether α_c reaches 0 for some physically accessible value of $\Delta v/\bar{v}$, and also the nature of the phase boundary in the $(t_\perp, \alpha)$ plane separating a coherent ("deconfined") 2D state from the state of "confined coherence", $({\rm 1D})_{\rm NFL}\otimes({\rm 1D})_{\rm incoh}$.

V. INTERPRETATION OF THE INCOHERENT RESPONSE

We are really interested in the nature of the true groundstate of the strongly interacting problem at strong coupling and finite $t_\perp$. Our dynamical calculation is meant to tell us something about this with as few assumptions and as much sensitivity to the possible breakdown of coherence as possible. This is necessary since the true ground state is not accessible by any controlled calculation at this time [57] and, as we have seen, more conventional calculations are difficult to interpret for questions of coherence.

We believe that the coherence question is crucial for the simple reason that the splitting of the Fermi surface for two chains or the formation of a two dimensional Fermi surface for infinitely many chains are specific manifestations of interference effects between histories which involve particles hopping between chains. This is most easily seen for the case of free particles where calculations can be carried out exactly and the single electron self-energy responsible for splitting the Fermi surface clearly relies on interference between paths that hop between different chains. On the other hand, our *definition* of a phase with incoherent hopping is that such effects are *completely non-existent* in such a phase on all energy scales small compared to the high energy cut-off of the theory. There can therefore not be any split Fermi surface (for two chains) or two dimensional Fermi surface (for many chains) in this case, and, since the shape of the Fermi surface is a low energy property of the system, we are indisputably dealing with a different fixed point from any phase in which split Fermi surfaces are expected. Coherence, or its absence, is crucial to the discussion since *previous proposals for fixed points with finite $t_\perp$ implicitly assume it to be present* by taking the dominant action of the single particle hopping to be the formation of symmetric and antisymmetric fermion operators, with corresponding Fermi surfaces. There has been, up to now, no attempt to justify this assumption, and in fact our work is really an attempt to provide that justification *and to attach proper limits to it.*

The approach to coherence we have followed parallels that in the TLS [31] problem; we have looked for interference effects in a particularly natural dynamical calculation where they would manifest themselves as damped oscillations in $\langle \delta N(t) \rangle$. If they are *completely* absent there, as they are for $\langle \sigma^z(t) \rangle$ at $\alpha = 1/2$ in the TLS, then we are interpreting this as demonstrating that they are completely absent in general. The assumption that the vanishing of the interference in the quantity we calculate signifies the loss of coherence and the inability to observe interference as defined by other measures, such as the presence of split Fermi surfaces, is a plausible assumption, which is identical to the usual assumptions made about macroscopic quantum coherence [31]. Further, there is no evidence of any kind which supports the contrary assumption that, for reasons unstated, a relevant $t_\perp$ should always be taken to act coherently and allow interference effects. In fact, on the experimental side, as we will see in Section VIII, there is extremely strong experimental evidence for a low temperature phase in which no interference effects for out of *plane* paths can be observed in the magnetotransport measurements on the organic conductor $(TMTSF)_2PF_6$. In a similar way, as discussed in the next Section, experimental evidence points to a phase (albeit bounded below in temperature by T_c) in the cuprate superconductors where interplanar transport is incoherent in a nontrivial way (*i.e.* the incoherence is not the result of inelastic scattering of any conventional type). These observations fit in perfectly with a trivial two dimensional generalization of our proposal, while it is difficult to see how they could be reconciled with

any other currently existing proposal for transport in anisotropic systems.

Returning to the specific interpretation of our calculation, we first note that for all the models which we understand (coupled free particles or Fermi liquids) and which have coherent ground states (the states are built up out of quasiparticle operators that have a well defined $k_\perp$ and energies which are functions of $k_\perp$), our dynamical calculation turns up a coherent response with oscillation frequency essentially equal (for two chains) to the splitting of the symmetric and antisymmetric quasiparticle combinations. Consequently, we propose to interpret the frequency of any oscillations which are present as the effective splitting of the symmetric and antisymmetric Fermi surfaces, i.e. the transverse bandwidth. We have already seen in Sections III and IV that our approach, with this interpretation, correctly describes all of the models where coherence is known to be present. Also, for models where $t_\perp$ is renormalization group irrelevant, we find no oscillations *and* a convergent long time response, and no oscillations, so that again our calculation and interpretation capture the correct physics.

Significantly, our calculation also exhibits a third regime directly analogous to the incoherent regime of the TLS. We interpret this regime, in which there is no sign of oscillations in $\langle \delta N(t) \rangle$ but $t_\perp$ is still relevant, as one of incoherence, in the sense that interference effects cannot be observed between histories in which $t_\perp$ acts. This parallels the interpretation of the corresponding TLS phase in the problem of macroscopic quantum coherence. This incoherence results because the only term in our Hamiltonian which connects states with different numbers of particles in a given liquid acts in a non-degenerate way: it connects predominantly states with energies differing by amounts large (in fact of order the cut-off in the theory) compared to the matrix element to those states. Under these circumstances, it should be impossible to have coherence in the sense of measurable interference between histories in which $t_\perp$ acts. A split Fermi surface should therefore not occur and the system should not have a groundstate which is built up out of symmetric and antisymmetric combinations of fermion operators. At the very least, no energy would be gained by forming such combinations since the matrix elements to the states with low enough energies to be mixed in coherently are vanishingly small (otherwise we should have seen *finite*, if heavily damped oscillations, in $\delta N(t)$ [58], signaled in our calculation by short time coherence). Further, such states are defined in terms of phases which are only meaningful if interference effects of the unobservable type are considered. We therefore regard the action of $t_\perp$ as qualitatively different in this phase than in the coherent phase.

Parenthetically, we should note that the physics we are interested in is the coherence of $t_\perp$. Our discussion throughout has neglected and will neglect operators such as interchain magnetic interactions generated by the renormalization of $t_\perp$. In particular models, these operators may in the end dominate the physics and determine the low energy behavior of the systems. However, we are certainly free to consider sufficiently general models that the coefficients of these other operators can be tuned so as to make them unimportant. From this consideration, it is clear that *in principle* these other operators are unimportant for the question of the existence or non-existence of the incoherent fixed point we are advocating. The question of whether they are in practice important for any specific model is beyond the scope of this paper, except for the brief discussion of Sec. VI.

Returning to our main theme, we believe that dynamical incoherence excludes the notion of Bloch states with a definite $k_\perp$. The reason for this is that, if dynamical incoherence

implies a more general incoherence such that interference effects are absent between histories that involve the action of $t_\perp$, then we have no way of comparing the phases of creation operators acting in different liquids and the whole notion of a Bloch state is meaningless for a discussion of the dynamics. In fact, Bloch states are exactly analogous to Schrödinger's cat: there is no point in writing down $|\text{alive}\rangle + e^{i\theta}|\text{dead}\rangle$ because the alive and dead histories will decohere before any interference can be observed, likewise for $\psi_1^\dagger + e^{i\theta}\psi_2^\dagger$ the two fermion operators will decohere too fast for θ, or equivalently, $k_\perp$ to have any meaning.

Certainly, if a two chain system were well described by nearly non-interacting symmetric and antisymmetric Bloch states, then coherent dynamics would result. The absence of such coherence in the dynamics therefore excludes this possibility. It is much more difficult to make definitive statements about what the dynamics would be like should the system be well described by, for example, symmetric and antisymmetric Luttinger liquids. Nonetheless, we are unaware of any reason to expect that the damped oscillations one expects in the dynamics of $\langle \delta N \rangle$ could in this case completely vanish despite the manifest coherence of the groundstate. After all, the oscillations are an interference effect and such effects ought to be observable under any circumstances where Bloch states are meaningful. In the absence of coherence in $\langle \delta N(t) \rangle$, we therefore believe that the groundstate cannot be thought of as built up out of operators creating and destroying Bloch states of quasiparticles with definite transverse momentum.

Incidentally, it is clear that no finite amount of damping implies incoherent transport. If interference effects are observable at all, then there is no barrier to the formation of Bloch states with definite transverse momentum and a higher dimensional Fermi surface (although the transverse bandwidth may be more heavily renormalized than suggested by a naive RG calculation). If a higher dimensional Fermi surface forms and the system crosses over to a Fermi liquid, then there will be coherent transport in the perpendicular direction because the scattering of quasiparticles always vanishes as the Fermi surface is approached. The quasiparticles therefore have well defined $k_\perp$ and $v_\perp$ and transport can be coherent transverse to the chains. This should be kept in mind when considering approximate approaches to calculating the single particle Green's function, since these, in general, result in predictions of non-vanishing scattering as the Fermi surface is approached; this must be an artifact. Since the dynamical quantity we are calculating does not directly involve the true ground state, this argument does not imply that the damping of its oscillation is an artifact (it isn't), but it does make clear that the damping, *if finite*, is not clearly meaningful for transverse transport. On the other hand, the smallness of the oscillation frequency, and certainly its vanishing, are very meaningful.

When we find a coherent response, describing damped oscillations with some frequency, it is reasonable to believe that the true groundstate should be constructed by first making single particle Bloch states with definite $k_\perp$ and then considering the interactions of these states. In higher dimensions, this should lead us back to Fermi liquid theory (or at the very least a higher dimensional Fermi surface), although the case of a finite number of coupled liquids may be subtle (see Finkelshtein and Larkin [35,59]). We believe, as previously stated, that the frequency of the damped oscillation in $\langle \delta N(t) \rangle$ should in general be equal to the transverse bandwidth, up to finite factors of order one. In fact, we saw that this was the case for coupled free Fermi gasses and Fermi liquids. If Luttinger liquids were coupled with a large enough $t_\perp$ or small enough α, $v_\rho - v_\sigma$, *etc.* that damped oscillations resulted, we

would identify the oscillation frequency with the transverse bandwidth. This should be $O(t_\perp^R)$ and in that sense corresponds closely with the proposals made by other authors for the transverse bandwidth [35], however in our case the frequency can reach zero in spite of the renormalization group relevance of $t_\perp$. In this case, the naively $O(1)$ prefactor between $t_\perp^R \sim t_\perp^{1/(1-2\alpha)}$ and the oscillation frequency is vanishing and, by implication, there should be no transverse bandwidth (although particles can clearly still move incoherently between chains, since $t_\perp$ is not irrelevant).

Since the transverse bandwidth specifies the splitting (warping) of the Fermi surface expected for two (many) chains, we propose that this splitting is in effect the *order parameter* for the coherence/incoherence transition. It is the most natural low energy example of an interference effect between histories involving interliquid hops, and, since the shape of a Fermi surface is an infinitesimal energy/infinite time property of the system, our proposed interpretation of the splitting/warping as an order parameter implies that the incoherent phase constitutes a truly distinct fixed point. This is different from the TLS problem where the coherent and incoherent regimes need not correspond to different fixed points since the marked distinctions between their short time behaviors need not persist to infinite times [60] since there is no corresponding long time measure of coherence. On the other hand, for fermions, the splitting or warping of the Fermi surface should be determined by the presence or absence of coherence, since the coherent oscillation frequency, if any, specifies the splitting. This splitting (warping) is a true long time property accessible in the measurement of asymptotic correlation functions, and therefore a qualitative change here requires a new fixed point.

The transition from coherence to incoherence is, however, not a thermodynamical one, but one in transport properties, more closely analogous to the Anderson metal-insulator transition in localization than to a thermodynamic phase transition. We therefore do not expect experimental signatures of it in specific heat, etc. but rather in transport data such as the resistivity. In particular, a substantial change in the perpendicular conductivity should occur at the transition, particularly in its frequency dependence, as we have discussed in Ref. [8].

In conclusion, the incoherence we are proposing is thus a truly new state of matter with novel properties arising from the absence of certain interference effects. Physically, we expect this to lead to host of unusual properties, including incoherent transport in one direction *all the way down to zero temperature in a pure system.* Fortunately for our rather radical proposal, it appears that this state has already been seen in $(TMTSF)_2PF_6$, as we shall shortly discuss.

VI. THE EFFECTS OF OTHER OPERATORS

Up to this point we have focussed exclusively on the nature of the single particle hopping between the Luttinger liquid chains; however, we know that one of the effects of higher order perturbation theory in $t_\perp$ is to generate other potentially relevant operators, such as magnetic superexchange between the chains and so on. How important are these effects for our conclusions?

Let us begin with the question of whether the dominant instability of the uncoupled

chains fixed point is to single particle hopping or to some other operator generated by the hopping at higher order. We discuss here only the case of fermions with spin. The lowest order renormalization group equations for $\lambda_{\rm single}$, the coupling of the single particle hopping operator, and the coupling constant, λ_O, of a generic operator, O, generated at higher order read:

$$\frac{\partial \lambda_{\rm single}}{\partial \ln \Lambda} = (d_t - 2)\lambda_{\rm single} \tag{117}$$

$$\frac{\partial \lambda_O}{\partial \ln \Lambda} = (d_O - 2)\lambda_O + \alpha_O \lambda_{\rm single}^p \tag{118}$$

where α_O is a coefficient of order $\Lambda^{-(p-1)}$ which depends on O, d_t is the scaling dimension of $t_\perp$, d_O is the scaling dimension of O and p is the order at which the operator is generated. In practice, the first operators to supersede the single particle hopping in importance are two body operators, for which $p = 2$. For repulsive interactions, the most relevant of these operators couples the $2k_F$ components of the spin operators on the two chains and we have:

$$d_t = \frac{1}{2} + \frac{1}{4}\left(K_\rho^{-1} + K_\rho\right) \tag{119}$$

$$d_O = 1 + K_\rho \tag{120}$$

The spin coupling is more relevant than the single particle hopping for all $K_\rho < 1/3$, however, to determine if it provides the leading instability we need to consider the initial conditions for the renormalization group flows:

$$\lambda_{\rm single}^{\rm initial} = t_\perp \tag{121}$$

$$\lambda_O^{\rm initial} = 0 \tag{122}$$

In this case, the criterion, in the limit of small $t_\perp$, for O to dominate, which one obtains by integrating the renormalization equations until one of the couplings is of order the cutoff, Λ, is:

$$d_O < 2(d_t - 1) \tag{123}$$

or, equivalently,

$$K_\rho < \sqrt{5} - 2 \tag{124}$$

We have obtained this result independently but it is equivalent to that of Boies, *et al.* [35].

Recall that, according to our arguments, the action of $t_\perp$ is incoherent for $\alpha > \alpha_c < 1/4$. This involves $K_\rho < K_\rho^c > 2 - \sqrt{3}$. Since $2 - \sqrt{3} > \sqrt{5} - 2$, there is guaranteed to be a region where the single particle hopping is simultaneously incoherent and the leading instability of the uncoupled chains fixed point. There is therefore no obstacle to realizing incoherence due to the two particle instabilities of the uncoupled chains fixed point. In fact, one could consider more general models of coupled chains where a *ferromagnetic* superexchange was added by hand, and then tune the coupling of the superexchange to cancel the coupling generated by $t_\perp$ and thus reach smaller values of K_ρ with the single particle hopping still

dominant. Consequently, it should be clear that the effects of these operators are model dependent. However, it is worth noting that even in the simplest coupled chains model, fine tuning is unnecessary and incoherent hopping is the dominant instability over a finite region of parameter space.

The dominance of the incoherent, single particle hopping is not sufficient to establish that a fixed point will be realized with purely incoherent hopping between the Luttinger liquids and no other couplings. This is because the incoherent fixed point itself may be unstable to some other operators. The stability analysis of the incoherent fixed point is beyond our present understanding of the problem, however, there are several points we wish to emphasize. First, the instability of the *uncoupled chains fixed point* to two body operators, which has been emphasized in many other studies of this problem [35], is completely irrelevant to the question of the stability of the *incoherent fixed point*. If the leading instability of the uncoupled chains fixed point is to single particle hopping, and we are in a regime where the hopping is incoherent, it is the stability or instability of the incoherent fixed point which ultimately determines the low energy physics, and the fact that the isolated chains fixed point is always unstable to some two body operator has no bearing.

Second, while, for any particular model, the stability of the incoherent fixed point may be very important, unless the incoherent fixed point is unstable to infinitely many operators, one can, in principle, find a sufficiently general model to tune the coefficients of all operators relevant about the incoherent fixed point so that these coefficients vanish. In that case the incoherent fixed point will be realized in the low energy limit. In light of this, it seems clear to us that what we have done is sufficient to demonstrate that *the incoherent fixed point does exist*; whether it is stable for a specific model, such as that of two or many chains coupled only with a transverse hopping, is a different question, and one which requires further study.

Finally, the concept of incoherent hopping clearly generalizes straightforwardly to the coupling of planes of strongly interacting fermions with non-Fermi liquid groundstates, and in this case one would expect that the incoherent fixed point would be substantially more stable than for the one dimensional case since two dimensional non-Fermi liquids need not exhibit the nesting related instabilities of one dimensional Luttinger liquids. In fact, the experimental case appears to be that the incoherent state in $(TMTSF)_2PF_6$ is unstable only to superconductivity (see Sec. VIII), and becomes the true groundstate in a sufficiently strong magnetic field. In the cuprates (see Sec. VII), there appear to be additional instabilities, perhaps due to finite strengths for the intra-bilayer couplings in bilayer compounds, and the superconducting instability is also clearly present. Nonetheless, many of the finite temperature and finite frequency properties appear to be the result of the proximity of the materials to the incoherent fixed point.

In summary, the inclusion of other operators beyond the single particle hopping poses no problem, in principle or in practice, for our arguments that the leading instability of the uncoupled Luttinger liquid fixed point can be to incoherent, single particle hopping. In this case, there should, at the very least, be a sizable region in energy/temperature where the properties of the system will be dominated by the underlying incoherent fixed point. The actual realization of this fixed point in the truly low energy limit is clearly possible in principle, while the realization in practice of the natural higher dimensional analog has, we believe, been demonstrated experimentally (see Sec. VIII).

VII. NORMAL STATE C-AXIS CONDUCTIVITY IN THE CUPRATES

The *qualitative* anisotropy exhibited by the cuprate superconductors in the normal state implies that they should be described by some fixed point of the renormalization group that is itself qualitatively anisotropic. However, the photoemission data provide strong evidence in favor of the renormalization group relevance of the interlayer single particle hopping. This implies that the correct low energy fixed point is in fact three dimensional, yet the transport data require a qualitative anisotropy. To reconcile these two features of the experimental data requires a non-trivial *three dimensional* alternative to the usual Fermi liquid fixed point. Note that the perturbative stability of Fermi liquid theory in three dimensions does not rule out the existence of such a fixed point; nor does the instability of the $t_\perp = 0$ fixed point. In general, when the weak coupling (here, $t_\perp = 0$) fixed point is unstable and there exists a stable strong coupling fixed point (here, three dimensional Fermi liquid theory with finite $t_\perp$), one expects that the flows will carry the effective Hamiltonian continuously from the weak to the strong coupling fixed point, however, this need not necessarily be the case since alternative, stable, strong coupling fixed points may exist. The primary goal of this paper is to propose such an alternative fixed point in which $t_\perp$ has not renormalized to zero and yet a qualitative anisotropy remains at the fixed point. The fixed point we propose is one of "confined coherence", at which the interlayer hopping is totally incoherent.

With this in mind, we will now present a more detailed discussion of the c-axis transport in the HTSC's. A comprehensive survey by Cooper and Gray of the data up to 1993 can be found in Ref. [61] to which the reader is referred for greater detail and a significantly more complete bibliography.

A. Boltzmann Theory

We begin by considering the very simplest estimates of conductivity in a highly anisotropic (2+1)-D metal with a single conduction band described by a tight-binding Hamiltonian

$$H = -2t\left[\cos(k_a a) + \cos(k_b b)\right] - 2t_\perp(k_a, k_b)\cos(k_c c) \tag{125}$$

We have included $\mathbf{k}$-dependence of $t_\perp$ to mimic more accurately the results of electronic band structure calculations (*e.g.* for the cuprates, the form $t_\perp(k_a, k_b) = t_\perp[\cos(k_a a) - \cos(k_b b)]^2/4$ has been proposed [62]). Of course, such a model omits nontrivial many-body effects, and is an oversimplification of the band structure, however it will suffice for the points we wish to make.

When the anisotropy is large, $t_\perp/t \ll 1$, the Fermi surface is open. To simplify the analysis, we suppose that the Fermi surface at constant k_c is approximately circular (this is certainly true if the electron density is not too large). The Fermi surface is then a warped cylinder, and we may linearize the dispersion in the ab-plane, whence

$$E(k, k_c) = \hbar v_F(k - k_F) - 2t_\perp(\mathbf{k})\cos(k_c c) \tag{126}$$

where $\mathbf{k} = (k_a, k_b)$ and $k = |\mathbf{k}|$. The conductivity tensor derived from simple Boltzmann transport theory may be written as

$$\sigma_{\alpha\beta} = \frac{1}{4\pi^3} e^2 \int_{\partial\mathcal{F}} [v_\alpha v_\beta \tau](\mathbf{k}, k_c) \frac{dS_F}{|\nabla_{\mathbf{k}} E|}$$
$$\approx \frac{1}{4\pi^3} \frac{e^2}{\hbar} \frac{1}{v_F} \int_{\partial\mathcal{F}} [v_\alpha v_\beta \tau](\mathbf{k}, k_c)\, dS_F \tag{127}$$

The final line follows from the anisotropy assumption $t_\perp/t \ll 1$, and reflects the fact that on an open Fermi surface, despite the small dispersion along the c-axis, there is no $\mathbf{k}$-state with "small" velocity - all velocities are of order v_F.

The c-axis and a-axis conductivities are therefore

$$\sigma_c = \frac{1}{4\pi^3} \frac{e^2}{\hbar} \frac{1}{v_F} \int_{\partial\mathcal{F}} [v_\perp(\mathbf{k})]^2 \, \sin^2(k_c c)\, \tau(\mathbf{k}) \tag{128}$$

$$\sigma_a = \frac{1}{2} \frac{1}{4\pi^3} \frac{e^2}{\hbar} v_F \int_{\partial\mathcal{F}} \tau(\mathbf{k}) \tag{129}$$

where $v_\perp(\mathbf{k}) \equiv 2t_\perp(\mathbf{k})c$.

In the isotropic-τ approximation we arrive at

$$\sigma_c \approx \frac{1}{2} \frac{1}{4\pi^3} \frac{e^2}{\hbar} \frac{1}{v_F} \tau \, (2\pi k_F) \, \frac{2\pi}{c} \, \langle v_\perp^2(\mathbf{k}) \rangle$$
$$= \frac{1}{2\pi} \frac{e^2}{\hbar} k_F \frac{\langle v_\perp^2(\mathbf{k}) \rangle}{v_F^2} \left(\frac{v_F \tau}{c} \right) \tag{130}$$

and

$$\sigma_a \approx \frac{1}{2\pi} \frac{e^2}{\hbar} k_F \left(\frac{v_F \tau}{c} \right) \tag{131}$$

The relation $\sigma_c/\sigma_a \approx (v_\perp/v_F)^2$ is a quite general result for a metal of high anisotropy, whenever simple Boltzmann transport theory and the isotropic-τ approximation are valid.

B. Mott minimum metallic conductivity

A simple argument attributed to Mott [63] (and to Ioffe and Regel [64] in a different context) provides an estimate of the minimum conductivity a metal must have in order to justify a straightforward application of Boltzmann transport theory. For a metal with a spherical Fermi surface the simple result $\sigma = ne^2\tau/m$ may be rewritten as

$$\sigma = \frac{1}{3\pi^2} \frac{e^2}{\hbar} k_F \, (k_F l) \tag{132}$$

using $n = k_F^3/3\pi^2$ and $l \equiv v_F\tau$ is the mean free path. Putting $k_F \sim \pi/a$, the Mott-Ioffe-Regel limit, $\sigma^{\min}$, is obtained by setting $l/a = 1$

$$\sigma^{\min} \sim \frac{1}{3} \frac{e^2}{\hbar} \frac{1}{a} \tag{133}$$

The physics of this "minimum metallic conductivity" is that once the point is reached where the electronic mean free path, l, is of the order of an interatomic spacing, a, the whole

apparatus of Boltzmann transport theory based upon Bloch states of definite momentum becomes questionable.

Mott's argument may be suitably modified for an anisotropic metal using the results derived above. The minimum c-axis conductivity is obtained by putting $v_\perp \tau = c$ giving

$$\sigma_c^{\min} \sim \frac{1}{2\pi}\frac{e^2}{\hbar} k_F \left(\frac{v_\perp}{v_F}\right) \tag{134}$$

where $v_\perp$ is short for $\langle v_\perp^2(\mathbf{k})\rangle^{1/2}$. Similarly, $v_F\tau = a$ gives

$$\sigma_a^{\min} \sim \frac{1}{2\pi}\frac{e^2}{\hbar}\, k_F \frac{a}{c} \tag{135}$$

We now apply these results to $La_{2-x}Sr_xCuO_4$. Using $a \approx 0.4$ nm, $c \approx 0.8$ nm, the band structure estimate [65] $v_\perp/v_F \approx 1/5$ for $La_{1.85}Sr_{0.15}CuO_4$, and estimating k_F by $n = k_F^2/(2\pi)$ with $n \approx a^{-2}$, *i.e.* one electron per Cu, we find $\sigma_c^{\min} \approx 500\,\Omega^{-1}$ cm^{-1} and $\sigma_a^{\min} \approx 1200\,\Omega^{-1}$ cm^{-1} [66]. By examining the resistivity data for $La_{2-x}Sr_xCuO_4$ shown in Fig. 7 we see that at the lowest normal state temperatures accessible σ_c is well below $\sigma_c^{\min}$ in all but the most overdoped samples. In contrast, σ_a is always well above $\sigma_a^{\min}$. Moreover, there is something very wrong with the estimate $\sigma_a/\sigma_c \approx (v_F/v_\perp)^2 \approx 25$: for example, for the $x = 0.15$ sample, $\sigma_a(T)/\sigma_c(T) = \rho_c(T)/\rho_a(T)$ *increases* monotonically as T decreases, with $\sigma_a(T_c)/\sigma_c(T_c) \approx 700$!

At this point some remarks are in order. The most important is that standard Fermi liquid quasiparticle renormalizations do not alter the above results. To see why, consider that k_F is an invariant under such renormalizations, a consequence of Luttinger's theorem. At sufficiently low temperature, scattering is dominated by impurities, in which case $v_F\tau = l_a$ is an invariant. Finally, $v_\perp/v_F$ is an invariant because both $v_\perp$ and v_F are renormalized by the same quasiparticle renormalization factor, Z. Hence, both σ_c and σ_a are invariants. This is the generalization of the isotropic result (132) which again is easily seen to be robust to even the most severe Fermi liquid renormalizations. Quite generally, velocity reducing factors are exactly cancelled by density of state enhancement. It is this fact which makes the Mott argument so powerful. Although Mott originally developed the argument in the context of localization theory, we emphasize that it is a general argument indicating how large the conductivity of an electronic Fermi liquid *has* to be in order to justify the use of quasiparticle Boltzmann transport theory.

The second remark concerns the use of the isotropic-τ approximation. One could envisage a situation in which $\tau(\mathbf{k})$ was a strongly $\mathbf{k}$-dependent function, such that $\tau(k_c \approx 0) \gg \tau(k_c \neq 0)$, in which case σ_c could be significantly further reduced. Such a scenario could arise if planar defects, parallel to the ab-planes, were the dominant source of scattering [67]. In the case of $La_{2-x}Sr_xCuO_4$, however, there is no experimental evidence for such a scenario: instead, one would expect the dominant source of scattering to be from isolated Sr atoms which are randomly arranged throughout the sample and reside approximately 0.4 nm from the CuO_2 planes, giving at most a weakly $\mathbf{k}$-dependent mean free path.

Thirdly, it should be emphasized that, as discussed in the Introduction, the above theoretical considerations have been verified in the highly anisotropic (2+1)-D metal Sr_2RuO_4 [29]. The low temperature in-plane and c-axis resistivities, plotted as a function of T^2,

are shown in Fig. 8. The behavior is FL-like in all directions. A crude estimate of the anisotropy from band structure calculations [28] gives $\rho_c/\rho_{ab} \approx 100$ in reasonable agreement with the data which give $\rho_c(T \to T_c^+)/\rho_{ab}(T \to T_c^+) \approx 500$. Of crucial import is the fact that $\sigma_c(T \to T_c^+) \approx 2000\,\Omega^{-1}$ cm^{-1}, much larger than the Mott limit $\sigma_c^{\min} \sim$ a few hundred Ω^{-1} cm^{-1}. The conductivity at low temperatures is truly metallic in all directions. We believe that Sr_2RuO_4 is the most anisotropic Fermi liquid known to date.

A comparison of the dc-conductivity of $La_{2-x}Sr_xCuO_4$ and Sr_2RuO_4 therefore leads us to the conclusion that anisotropy alone cannot account for the anomalous c-axis transport in $La_{2-x}Sr_xCuO_4$. The behavior of $\rho_c(T)$ cannot be accounted for within a Fermi liquid framework; there is no plausible mechanism to lead to a c-axis scattering rate larger than $t_\perp$. We are therefore led to what we believe is the *only* way to understand the c-axis transport in the normal state of many of the HTSC's. This is to associate the anomalous c-axis conductivity with the fact that the in-plane electron liquid is not a Fermi liquid.

C. Further Empirical Evidence for Incoherence

There are other, equally compelling, experimental facts which are at odds with any Fermi liquid description of c-axis transport in the cuprates. The first is the frequency-dependent conductivity, $\sigma_c(\omega)$, data for which are shown in Fig. 9 for $La_{2-x}Sr_xCuO_4$ [22]. For all but perhaps the most overdoped sample ($x = 0.3$) there is no plausible way to argue for a Drude, or generalized-Drude, contribution. The same is true for $YBa_2Cu_3O_{6+x}$ (Figs. 10, 11), even the "optimally doped" $YBa_2Cu_3O_7$ where the "linear-T" [68] behavior of ρ_c has often been interpreted as a sign of metallic c-axis conduction. As the data of Schützmann *et al.* [69] show (Fig. 12), a fit of a Drude term σ_c^D to the low frequency $\sigma_c(\omega)$,

$$\sigma_c^D(\omega) = \left(\frac{ne^2}{m_b}\right) \frac{\Gamma(\omega)}{\Gamma^2(\omega) + \omega^2(m^*(\omega)/m_b)^2}$$

leads to a width $\Gamma(\omega \to 0) \sim 1500$K at T=100 K, much too large to account for either in a conventional way or in a way connected to the anomalous in-plane relaxation rate $\Gamma_{ab} \sim$ T. In fact, the raw data are very flat, or perhaps weakly rising with frequency, at all but the lowest and highest (of order an eV) frequencies.

Note that in discussing the dc-conductivity in the previous section, it was necessary to use estimates of the c-axis bandwidth from band structure calculations in order to estimate, for example, $\sigma_c^{\min}$. Within a Drude framework, the dc-conductivity is $\sigma_c^D(\omega = 0) = ne^2/(m_b\Gamma)$, so that we only obtain direct information about the combination $m_b\Gamma$. The important advantage of ac-conductivity experiments is that they enable one to disentangle the mass of the carriers from the relaxation rate. As a result, any attempt to explain away the poor dc c-axis conductivity in terms of a very small c-axis bandwidth is at odds with the $\sigma_c(\omega)$ data which exhibit spectral weight over a very wide energy range. Put another way, $\sigma_c(\omega)$ *cannot* be characterized by a narrow Drude conductivity of small weight. Were one willing to ignore the band theory estimates of $t_\perp$ one could have proposed such a scenario on the basis of the dc-conductivity data alone.

Again, the c-axis conductivity observed in Sr_2RuO_4 provides a stark contrast to the cuprates. Recent measurements by Katsufuji *et al.* [70] exhibit very beautifully the develop-

ment of a Drude peak in $\sigma_c(\omega)$ at temperatures below 100 K or so. Thus, Sr_2RuO_4 provides a very important control system to test predictions for transport in a highly anisotropic Fermi liquid.

Further experimental evidence in favor of c-axis incoherence can be found by considering $Bi_2Sr_2CaCu_2O_8$. This is a bilayer HTSC in which the *inter*bilayer hopping, $t_\perp^{\rm inter}$ is significantly smaller than the *intra*bilayer hopping, $t_\perp^{\rm intra}$. Reflectivity experiments show no indication of any c-axis Bloch-Boltzmann quasiparticles and the plasma edge is below 30 $\rm cm^{-1}$, corresponding to an effective mass anisotropy $m_c^*/m_{ab}^* \approx 100$. On the other hand, band structure calculations estimate $t_\perp^{\rm intra}$ to be about $100\, meV$. Quite generally, if $t_\perp^{\rm inter}/t_\perp^{\rm intra} \ll 1$, Fermi liquid theory predicts *two* Fermi surfaces split in energy by $\sim 2t_\perp^{\rm intra}$, with each exhibiting a c-axis dispersion $\sim t_\perp^{\rm inter}$. Therefore, even though the dc- and low-frequency conductivity may be poor, a strong absorption signal (albeit broadened by anisotropy of $t_\perp$ and lifetime effects) in $\sigma_c(\omega)$ should be observable at frequencies around $\sim 2t_\perp^{\rm intra}$, which corresponds to an inverse wavelength in the 1000-2000 $\rm cm^{-1}$ range. However, reflectivity data [23] for $Bi_2Sr_2CaCu_2O_8$ provide no evidence of such absorption. The same considerations may be applied to other bilayer HTSC's, $YBa_2Cu_3O_{6+x}$ for example. Again, there is no evidence of an intrabilayer splitting.

In principle, the most direct probe to locate Fermi surfaces is photoemission. The most recent photoemission experiments [25] on $Bi_2Sr_2CaCu_2O_8$ exhibit just a *single* Fermi surface, within an experimental resolution of $\sim 10\, meV$. This correlates completely with the absence of any resonance in $\sigma_c(\omega)$ in the 1000 $\rm cm^{-1}$ range. Electrons are apparently unable to tunnel coherently from plane to plane even *within* a bilayer.

This is consistent with the situation *vis a vis* the c-axis conductivity in the single layer material $La_{2-x}Sr_xCuO_4$. The conclusion to draw is that in all superconducting samples of HTSC's, c-axis transport is an incoherent process and that there is no physically plausible way to modify FLT to account for this incoherence. To emphasize the latter point, we now briefly discuss the various scenarios for c-axis transport which have previously been proposed.

D. Theoretical Implications of Incoherence

We have demonstrated two key aspects of normal state c-axis transport in the cuprates: (1) it is incoherent (hence non-metallic), as evidenced by experimental data; (2) taking band structure estimates for anisotropy, such incoherence is incompatible with a Fermi liquid description. Apart from the suggestion of anisotropic localization (discussed and criticized below), there are really only two ways to attempt to theoretically understand this situation, the first being to retain a FL-like description, and effectively ignore the band theory estimates of anisotropy, the second being to ascribe the anomalous c-axis transport to a non-Fermi liquid origin.

We begin with the former. These "conventional" attempts [71,72] at explaining the incoherent c-axis conduction in the HTSC's assume, either explicitly or implicitly, that the interplane hopping rate is sufficiently small that the Mott limit (134) is *not* violated at low temperatures. The zero temperature conductivity is then very small, but nonetheless metallic. One can then advocate various types of inelastic interplanar hopping processes,

which are additive to the conductivity, to generate a $\rho_c(T)$ which is a decreasing function of temperature. The problem with such an approach, though, is that it fails to address its very starting point, namely why the effective $t_\perp$ is so much smaller than the band theory prediction (in, say, $La_{2-x}Sr_xCuO_4$). Again, one cannot use standard band-narrowing effects from FLT, for these are *isotropic* in the sense that both t and $t_\perp$ are renormalized by the same factor, which therefore leaves $\sigma_c^{\min}(\omega)$ unchanged. In essence, these proposals demand some mechanism for an *anisotropic* downward renormalization of the bandwidths, the downward renormalization being required to be most severe in $t_\perp$. However, there is no such mechanism within FLT. From the experimental point of view, we again emphasize that the frequency dependent conductivity, $\sigma_c(\omega)$, has weight over a *very* broad frequency range, and is generically an *increasing* function of frequency, which at low temperature is completely incompatible with these conventional approaches.

If one accepts that the band theory estimates of $t_\perp$ are not wildly inaccurate, then one is forced to look elsewhere for an explanation of the poor c-axis conductivity. Within a simple Bloch-Boltzmann Fermi liquid framework there is really only one other parameter to play with, namely the relaxation time, $\tau(\mathbf{k})$. We have already mentioned above that we do not believe there is a plausible way to argue for a strongly anisotropic $\tau(\mathbf{k})$ (such a strong anisotropy is absolutely essential in the scenarios of Refs. [67] and [72]). A related proposal, however, has been put forward by Kotliar *et al.* [73]. These authors advocate a regime of "anisotropic localization" where the localization length ξ is sufficiently anisotropic that (for all temperatures above T_c) $\xi^{ab} > l_i^{ab}$ but $\xi^c < l_i^c$. Here l_i is the inelastic scattering length. Since effects of weak localization are only seen when $\xi < l$, the claim is that in such a regime, localization behavior will be observed in ρ_c but not in ρ_{ab}.

From a theoretical point of view, the problem with this suggestion is that one again must advocate a highly anisotropic random potential. The extreme case is that of unidirectional randomness, *i.e.* planar disorder, which will lead to states which are localized along the c-axis (since a 1D random potential will always localize) but extended in the ab-plane [67]. But, as pointed out earlier, in $La_{2-x}Sr_xCuO_4$ for example, there is no reason to believe that the disorder introduced by the Sr ions is anything like planar. Rather, it is almost isotropic in which case the appropriate localization problem to consider is that of quasiparticles with an anisotropic mass moving in an isotropically random potential. This problem has been studied by several authors [74,75] with the conclusion that localization occurs *simultaneously* in all directions. At non-zero temperature, any inelastic processes which cut off localization do so in a proportional way, in the sense that $\delta\sigma_{ab}/\sigma_{ab}^D = \delta\sigma_c/\sigma_c^D$ where $\delta\sigma$ denotes the departure from the Drude result σ^D. In particular, the temperature dependence of the conductivity will be the same in all directions, and one will not observe anisotropic localization. As Anderson has remarked [26], the physics here is that weak localization requires coherent backscattering of quasiparticles, and loss of coherence due to inelastic scattering in any one direction implies loss of coherence in all [76].

From an experimental point of view the anisotropic localization scenario is even less tenable because the virtually flat low frequency behavior of $\sigma_c(\omega)$ is completely at odds with weak localization.

A somewhat different scenario has been proposed by Kumar and Jayannavar [77] to account for HTSC samples which exhibit a $\rho_c(T)$ which *increases* with temperature. At temperatures such that the in-plane transport rate $\tau_{ab}^{-1} \sim k_B T$ is larger than $t_\perp$, one expects

to enter a regime where c-axis transport is rendered incoherent due to in-plane "dephasing". Here, the in-plane scattering is predominantly inelastic so that the phase coherence of an electron is destroyed at a rate faster than the rate of interplane hopping. In this regime ρ_c/ρ_{ab} can be arbitrarily large, but temperature independent. As such, this picture will only work at high temperatures, the lower temperature cutoff being roughly $t_\perp$. For $La_{2-x}Sr_xCuO_4$ (except highly overdoped) the $\rho_c(T)$ data set this cutoff at a few hundred degrees, in agreement with the band structure estimate $t_\perp \sim 500$ K. This scenario *cannot* account for a monotonically increasing c-axis resistivity (as temperature is lowered) coexisting with a monotonically decreasing ab-plane resistivity.

Having ruled out a FL description of the c-axis transport, we are forced to consider a NFL one. Unlike in the FL case where the interliquid hopping operator $t_\perp$ is always a relevant operator, in a NFL $t_\perp$ can be relevant, marginal or irrelevant depending upon the specifics of the NFL. However, as already discussed in the Introduction, the observation of sharply peaked spectral functions in ARPES experiments on $Bi_2Sr_2CaCu_2O_8$ and $YBa_2Cu_3O_{6+x}$ is *incompatible with an irrelevant* $t_\perp$. It is this fact which rules out a purely 2D description of the cuprates. To our knowledge, the only NFL approach to c-axis transport other than ours is that based on t-J model gauge theories. The most recent is the work of Nagaosa [78], and Lee and Wiegmann [79], where it has been argued that at temperatures $T \gg T_{\rm BE}$ (where $T_{\rm BE}$ is the holon condensation termperature) c-axis conduction is incoherent, and has temperature dependence $\sigma_c \propto xT^{1/2}$. In this regime, the electron may be treated as a convolution of essentially free holons and spinons, with the result that $t_\perp$ is a marginal operator. While the electron spectral function is dominated by its incoherent part, there is a "quasiparticle" part of weight x (the hole concentration), and width $\Gamma_{\rm qp} \sim (J\,T)^{1/2}$. The Schrieffer tunneling formula is used to calculate σ_c, which is a self-consistent procedure if $t_\perp \lesssim \Gamma_{\rm qp}$, *i.e.* $T \gtrsim t_\perp^2/J$. The source of the incoherent σ_c is therefore simply the large $\Gamma_{\rm qp}$, which is similar in flavor to the phenomenology of Kumar and Jayannavar [77]. Of crucial import, however, is that the gauge model calculation is only valid for $T \gtrsim T_{\rm BE} \sim xJ$, where J is the Cu-Cu antiferromagnetic superexchange, $J \sim 1500K$ in the cuprates. In fact, as explicitly shown in [78], there is a quasiparticle contribution to σ_c of the form $\sigma_c^{\rm qp} \sim x^2\,T^{-1/2}$ which starts to dominate over the incoherent contribution at a temperature $T \sim T_{\rm BE}$, and if this result was naively extended to the low temperature limit it would ultimately lead to a metallic c-axis conductivity $\sigma_c \sim x^2\,T^{-1/2}$. However, the calculation is only justified for $T \gtrsim T_{\rm BE}$, and in the gauge model $T_{\rm BE}$ is supposed to correspond to the superconducting transition temperature T_c. The claim then is that the metallic σ_c is not observed experimentally because of the onset of superconductivity. For the same reason, the Drude term predicted in $\sigma_c(\omega)$ (coming from the quasiparticle contribution) would not be resolvable above T_c due to the broadening from $\Gamma_{\rm qp}$. To summarize, the picture is that in the normal state the quasiparticle width due to thermal broadening is sufficiently large compared to the interlayer hopping rate as to render interplanar transport incoherent. Unlike the phenomenology of Kumar and Jayannavar, however, the gauge model is capable of producing a c-axis resistivity that increases as the temperature is lowered (provided $T > T_{\rm BE} \sim T_c$).

There are several criticisms of this approach. They all essentially stem from the same point, namely the validity of application of the Schrieffer tunneling formula. As mentioned above, one expects this to be valid if $t_\perp \lesssim \Gamma_{\rm qp}$, *i.e.* $T \gtrsim T_{\rm tunn} = t_\perp^2/J$. In $La_{2-x}Sr_xCuO_4$ the

band theory estimate of $t_\perp \sim 500$ K gives $T_{\rm tunn} \sim 200$ K, well above T_c. On the other hand, one might argue that the true regime of validity is nontrivial to determine, that $t_\perp \lesssim \Gamma_{\rm qp}$ is a conservative overestimate, and the true regime may extend to somewhat lower temperatures. Nevertheless, the criterion for validity of the tunneling formula raises its head when one tries to compare the calculation of σ_c with data. For simplicity, we shall confine ourselves again to $La_{2-x}Sr_xCuO_4$. Consider the result of [78]:

$$\sigma_c \sim e^2 \left(\frac{t_\perp}{J}\right)^2 \left[ax \left(\frac{J}{T}\right)^{1/2} + b\left(\frac{T}{J}\right)^{1/2}\right] \tag{136}$$

While we appreciate that it is unreasonable to demand a detailed agreement of Eqn. (136) with experimental data, there are several points worth noting. Firstly, at high temperature Eqn. (136) predicts $\sigma_c \sim T^{1/2}$, yet the experimental behavior at all doping levels is that σ_c *decreases* with T. Secondly, and of greater import, the predicted doping dependence is *much weaker* than that exhibited by the data. For example, Eqn. (136) predicts at most a factor of 4 ratio of the dc conductivity of $x = 0.1$ and $x = 0.2$ $La_{2-x}Sr_xCuO_4$, while the data exhibit a ratio as high as 40. In order to avoid this situation, Lee and Wiegmann [79] introduce a strongly doping dependent hopping rate, *i.e.* $t_\perp = t_\perp(x)$. For example, they take $t_\perp(x = 0.16) \sim 30$ K and $t_\perp(x = 0.3) \sim 200$ K. The argument is circular, however, since the $t_\perp$ estimates are taken from the *data*, whereas the $t_\perp$ in (136) should be taken from the band theory, in which case the doping dependence would be nowhere near as large. Moreover, Lee and Wiegmann argue that the tunneling formula is invalid in the overdoped regime, except at high temperature ($T \gtrsim 500$ K), which again raises the issue of the estimate of $T_{\rm tunn}$, and whether the use of the tunneling formula is ever valid in the range where experiments are performed, given the large band theory estimate of $t_\perp$. We note that the use of a strongly doping dependent $t_\perp$ is common to *all* of the theoretical models for c-axis transport discussed above, because the data require a large variation of σ_c with x. The only exception is the scenario we propose, where there is potential for a strong variation in σ_c with doping due to the complete lack of interplanar coherence even at $T = 0$.

To conclude, while the gauge model calculation and our proposal of "confined coherence" have common elements, most notably the use of a NFL in-plane state and the fact that incoherence results from a "broad" spectral function, we emphasize that in the gauge model picture the broadening is thermally induced, while in our picture incoherence is a property of a truly $T = 0$ state, due to a "broadening" of the spectral function which is not a thermal, nor an impurity, broadening. In fact, in our picture the "broadening" is not even of Lorentzian type. These two theoretical approaches to c-axis transport are therefore logically distinct.

Taken together, the ARPES data and the c-axis transport data *require a theoretical framework in which interplanar hopping is relevant but incoherent*. This is precisely the physics of "confined coherence", a specific example of which is afforded, as shown in the previous sections, within the context of weakly coupled Luttinger liquids. We believe that this is the most promising framework within which to understand the anomalous c-axis transport in the cuprates.

E. Summary

Upon examining the very large amount of data now available on c-axis transport in the cuprates and comparing it to theoretical expectations one is forced to accept that there is a very real anomaly. Moreover, there is really nothing physically reasonable one can do within a FL framework to account for the anomalous behavior. We have emphasized the importance of transport measurements in the highly anisotropic (2+1)-D metal Sr_2RuO_4 which are in accord with what is expected from the application of FLT to the predicted band structure. There is a striking difference between σ_c in Sr_2RuO_4 and σ_c in $La_{2-x}Sr_xCuO_4$. Our central point is that the one ingredient missing from conventional attempts to understand c-axis transport in the cuprates is the simple fact that they are *not* conventional metals: the in-plane liquid is *not* a FL. Disorder might be a complication, but it alone cannot account for the anomalous physics. In the absence of any plausible alternative capable of accounting for the universal fact of incoherent c-axis transport, the simplest problem relevant to the issue is that of coherence/incoherence of single particle hopping between NFL's (without disorder). We believe that the physics of "confined coherence" exhibited in the model of coupled 1D Luttinger liquids, when suitably generalized to coupled 2D NFL's, is the correct paradigm within which to be able to even qualitatively understand the anomalous c-axis transport in the cuprates. A first attempt at making semi-quantitative contact with experiments has been previously presented [8].

VIII. INCOHERENCE IN THE ORGANIC CONDUCTOR $(TMTSF)_2PF_6$

The original motivation for the proposal of the incoherent $t_\perp$ fixed point was the experimental data on the cuprate superconductors which conflicted with the predictions of both three dimensional Fermi liquid theory and theories in which the transverse hopping was irrelevant, however the theoretical arguments we have advanced for the potential incoherence of hopping between Luttinger liquids are quite general and other strongly interacting, highly anisotropic materials are also candidates for incoherent behavior. In fact, we believe that the organic conductor $(TMTSF)_2PF_6$ is very close to having incoherent single particle hopping in one direction and can be driven by an appropriate external perturbation into the incoherent regime. The experimental evidence supporting this contention is both extensive and strong as we will now discuss.

We begin with a brief discussion of the properties of the relevant material. $(TMTSF)_2PF_6$ is an organic conductor composed of long conducting chains of the organic molecule TMTSF stacked into planes which are separated by inorganic PF_6 anions. The resulting three dimensional structure is triclinic and its transport properties are highly anisotropic with tight binding bandwidths of $\sim 1\ eV$ in the main chain direction (hereafter a), $\sim 0.1\ eV$ in the other in-plane direction (hereafter b) and $\sim 0.003\ eV$ in the out-of-plane direction (hereafter c). These hopping integrals are so anisotropic that the material has a quasi-one dimensional Fermi surface with a pair of slightly warped Fermi sheets for its single conduction band. This band would be quarter filled except for a dimerization along the main chain direction which results in $(TMTSF)_2PF_6$ having a half filled conduction band. The effects of interactions should be substantial in this material as naive estimates of the on site Coulomb interaction

repulsion energy is of order the largest bandwidth and the conduction band is half-filled; the material is therefore both strongly correlated and highly anisotropic.

$(TMTSF)_2PF_6$ exhibits an extraordinarily rich low temperature phase diagram as a function of pressure and applied magnetic field, including spin-density wave and superconducting phases as well as a "normal" metal phase, confirming that correlation effects are strong and observable in this material. Here we concern ourselves with the metallic phase which occurs under a few kilobars of pressure for temperatures high enough or magnetic fields strong enough to destroy the superconducting state. In this phase the magnetoresistance of the material is highly anomalous both in magnitude and in its dependence on the direction of the field. The unusual behavior of the magnetoresistance provides the main evidence for the realization of incoherence in this material and will be central to our discussion.

Figure 13 shows the magnetoresistance data of Kang *et al.* [32]. In these experiments the current is applied along the $\hat{a}$ direction (the most conducting direction and the direction in which the bandwidth is largest) while magnetic fields of various strengths are rotated in the bc plane. The magnetoresistance is generally increasing as the field is tilted away from the b direction, but sharp dips in the magnetoresistance occur for field orientations where the field parallels a real space lattice vector. These orientations of the field are referred to in the literature as "Lebed magic angles" after a proposal by Lebed [80] that there should be features associated with the field induced spin density wave state for magnetic fields directed along real space lattice directions. There have been previous attempts to account for the features in the magnetoresistance at these angles based on commensurability effects [81] and there are in general many reasons why there might be features in the magnetoresistance at these angles. However, as we will discuss, the magnetoresistance data have features which can only be accounted for by the transition of $(TMTSF)_2PF_6$ to a state in which coherence is lost for single particle motion out of the ab planes.

Noting that $(TMTSF)_2PF_6$ is strongly correlated and highly anisotropic, one might expect that the single particle motion in this material might be incoherent in either the b and c directions or only in the c direction. The interchain analysis of Sec. III and IV would apply directly to the former case, and the latter case is possible if the ground state of a single ab plane of $(TMTSF)_2PF_6$ had a non-Fermi liquid ground state with properties similar to those of a one dimensional Luttinger liquid (e.g. anomalous exponents, power law singularity in n_k at k_F rather than a step function, different spin and charge velocities). Experimentally, the evidence is that *in the absence of a magnetic field with a sufficient projection onto the direction perpendicular to the a and c lattice directions* the material is three dimensionally coherent; we will discuss the evidence for this coherence later. For now, we take this as given, but consider the possibility that the single particle hopping in the c direction is very nearly incoherent. In this case a magnetic field along the direction perpendicular to the a and c lattice directions should have very novel effects.

The reason for this is that a field in this direction introduces inelasticity into the hopping of electrons in the c direction. After making a Peierls substitution one obtains that an electron in one plane with crystal momentum k hops into a state with crystal momentum $k + \hat{a}eBl_c$ when hopping one lattice spacing, l_c, in the c direction. Here e is the electron charge, B is the magnetic and l_c is the c-axis lattice spacing. This momentum shift implies an energy shift $\Delta E \sim v_F eBl_c$ because the highly anisotropic nature of $(TMTSF)_2PF_6$ gives it a well defined Fermi velocity in the a direction. As we have argued, the question of whether

or not $t_\perp$ acts coherently or incoherently is essentially the question, for small $t_\perp$, of whether or not perturbation theory in $t_\perp$ is degenerate or non-degenerate and the added inelasticity helps to further lift the degeneracy of the states connected by the $t_\perp$ operator, enhancing the possibility for incoherence. This is analogous to the increase in incoherence predicted for the two level system problem in the case of non-degenerate levels [31]. The reason for commensurability effects in the magnitude resistance is clear in this scenario: if the field is directed along a real space lattice direction out of the *ab* plane then, for hops in that direction, there is no change in crystal momentum thus no inelasticity induced by the field and no enhancement of the incoherence. The material should then retain three dimensional coherence. In general, this should result in a decrease in the electrical resistivity compared to cases where B is not parallel to a real space lattice vector, since the higher dimensional state should have substantially less scattering. The changes in the resistance associated with this transition should be large (of order the resistance itself) as the transition is a qualitative one. The dips should be particularly pronounced for *c*-axis transport data such as those of Figure 14, in which data of Danner *et al.* [33] are shown for magnetoresistance for currents perpendicular to the *ab* plane as fields of various strength are rotated in the *bc* plane.

While incoherence offers a natural explanation for the dip features in the magnetoresistance and can account for the anomalous magnitude of the magnetoresistance itself, other explanations for some of the behaviors have been proposed [81]. Fortunately, there are a number of truly unique predictions of the incoherence theory which offer genuine and stringent tests. Additionally, there are several qualitative features of the data which are inconsistent with all other proposed theoretical explanations but completely compatible with the incoherence theory.

We first discuss the four novel predictions of the incoherence explanation for the magic angle effects which are distinct from other theories (some of these were discussed in [7]). The first of these is that there is a hierarchy among the magic angle dips. For a dip in the *a* axis magnetoresistance to occur for an orientation corresponding to a given real space lattice direction, three dimensional coherence must rely on hopping in that direction. Since the coherence of hopping in any direction out of the *ab* plane implies three dimensional coherence, this means that to observe a dip hops in all other directions out of the *ab* plane must be incoherent; the central dip for fields nearly in the $\hat{c}$ direction is therefore primary. If the field is too weak to disrupt the *c* axis hopping sufficiently strongly for higher order hopping integrals like that in the $\hat{b}+\hat{c}$ direction to become important, then no feature will be seen in the vicinity of the other magic angles. This is born out experimentally as shown by Figure 15 which depicts the field dependence of the magnetoresistance in the vicinity of the $\hat{c}$ and $\hat{b}+\hat{c}$ magic angles.

For all fields strong enough to destroy the superconducting state, there is a sharp angular dependence near $\hat{c}$ but the angular dependence near $\hat{b}+\hat{c}$ turns on after the field has reached a strength of about 1 Tesla.

The second prediction of the incoherence theory is that the feature at $\hat{b}$ is *not a magic angle dip*. No transition to three dimensional coherence occurs when the field parallels $\hat{b}$ (in fact quite the contrary-two dimensionality is maximal). The shape and field dependence of this feature should therefore be completely different from those occuring at the other magic angles. This difference between this feature and those occuring at the magic angles is very clear in the data of Figure 13 but has been ignored by all previous theories.

The third prediction of the incoherence theory concerns the shape of the magnetoresistance curve everywhere away from the magic angle dips. Our proposal is that for fields with a large projection along the direction perpendicular to $\hat{a}$ and $\hat{c}$ and with large projections along the directions perpendicular to $\hat{a}$ and $n\hat{b} + \hat{c}$ for small n, the field can combine with the interaction effects to render all single particle hopping out of the ab plane completely incoherent. If this occurs, then in a path integral calculation of any physical quantity, the phase associated with taking an electron around a closed loop that does not lie entirely in a single ab plane will be randomized by the incoherence of the interplane hopping. Thus only loops lying entirely in a single ab plane have well defined phases, and since the only effect of a magnetic field (at the level of a Peierls substitution where we have ignored the effects of the field on the Wannier functions themselves on which the tight binding description is based) is to a add a phase proportional to the flux enclosed to the weight of such paths, the magnetic field will effect only these paths. Since only the component of the magnetic field that is perpendicular to the ab plane contributes to the fluxes such paths enclose, *all physical properties will be independent of the other two components of the magnetic field.* In particular the magnetoresistance will depend only on the component of the field out of the ab plane. It is already clear from Figure 13 that the magnetoresistance for a field along $\hat{b}$ is independent of field strength for fields larger than 0.8 Tesla, satisfying this prediction. In Figure 16, we replot the data of Figure 13 as $R_{xx}(H) - R_{xx}(H \cdot \hat{c} = 0)$ versus $\vec{H} \cdot \hat{n}_{ab}$ to demonstrate the more general scaling: *the magnetorestance is only a function of the field perpendicular to the ab plane.*

Not only is $R_{xx}(H) - R_{xx}(H \cdot \hat{c} = 0) = f(\vec{H} \cdot \hat{n}_{ab})$, but $R_{xx}(H) - R_{xx}(H \cdot \hat{c}) \propto \sqrt{\vec{H} \cdot \hat{n}_{ab}}$! The fact there there should exist some scaling function f so that $R_{xx}(H) - R_{xx}(H \cdot \hat{c} = 0) = f(\vec{H} \cdot \hat{n}_{ab})$ is a unique prediction of the incoherence theory which the data dramatically confirm. The only alternative explanation for this would be for the single particle hopping out of the ab plane to be an irrelevant operator. This cannot be reconciled with the fact that the conductivity out of the ab plane is not insulating for an 18 Tesla field directed along $\hat{b}$ down to temperatures of 50 mK [82] and the fact that the magnetoresistance for currents out of the ab plane can, depending on pressure, change by less than a factor of two as the transition to planes without coherent coupling is made. Further, there exists no proposal for a mechanism by which a magnetic field along the direction perpendicular to $\hat{a}$ and $\hat{c}$ should enhance the irrelevance of hopping out of the ab plane in the metallic state of $(TMTSF)_2PF_6$. Thus the simple irrelevance of $t_\perp$ appears thoroughly implausible as an explanation for the observed scaling behavior.

The fact that the scaling function for R_{xx} is a square root is not at present understood and is not a prediction of the incoherence theory. It is compatible with the theory and totally incompatible with any three dimensional Fermi liquid explanation for the magnetoresistance of $(TMTSF)_2PF_6$. Likewise, the data shown if Figures 14 and 17 for the magnetoresistance out of the ab plane, while not directly predicted by the incoherence theory, support it very strongly. The magnetoresistance in this direction also scales, depending only on $\vec{H} \cdot \hat{n}_{ab}$, being proportional to $\left(\vec{H} \cdot \hat{n}_{ab}\right)^x$ where $x \approx \frac{3}{2}$.

The anomalous power law requires a non-Fermi liquid state, while the scaling requires a two and excludes a three dimensionally coherent state. It is particularly striking that, except for the magic angle behavior, the magnetoresistance is largest when the component

of the magnetic field out of the *ab* plane is largest *even though the current is in that direction* while the magnetoresistance saturates to complete field independence for fields along $\hat{b}$ once incoherence has been achieved *even though this field is perpendicular to the current.* Semiclassically, no saturation of the magnetoresistance would occur for this second field-current configuration.

The final prediction of the incoherence theory for magnetoresistance experiments for fields in the *bc* plane is that for weak magnetic fields the magnetoresistance should exhibit a maximum for fields along the direction normal to $\hat{a}$ and $\hat{c}$, rather than the minimum observed at $\hat{b}$ at higher fields. This should occur as a result of the central dip expanding as the field is reduced until the data have smoothly crossed over to a form appropriate for a Fermi liquid with an open Fermi surface. This is exactly what occurs [82]. This result is particularly dramatic since it demonstrates that there really is a transition from a state with essentially the expected Fermi liquid properties to a state with exactly the expected properties for a state with "confined coherence". This makes it clear that (1) it is the field which induces the incoherence, not disorder or finite temperature and (2) the behaviors in the "confined coherence" region really are incompatible with Fermi liquid theory, it is not that we have somehow considered an inappropriate Fermi liquid model.

The incoherence theory thus offers a natural, simple explanation for the magic angle magnetoresistance effects in $(TMTSF)_2PF_6$, making *four unique, strikingly confirmed predictions* . Moreover, the theory is consistent with all aspects of the data where no direct prediction is yet possible, even those aspects of the data which are incompatible with other theories proposed to date. Further, as we now discuss, an independently proposed test of the theory has been carried out by Danner, *et al.* [33] which probes the qualitative nature of c-axis transport directly.

In investigating the magnetoresistance of $(TMTSF)_2ClO_4$, a similar material to $(TMTSF)_2PF_6$, for currents perpendicular to the *ab* plane and fields in the *ac* plane close to the $\hat{a}$ direction, Danner *et al.* found sharp magnetoresistance resonances as shown in Figure 18.

They accounted for these effects with a quasiclassical theory in which the averaging over quasiclassical orbits of the velocity out of the *ab* plane is more effective for certain field orientations than for others [83] and argued that these resonances can be used to probe the bandwidth of $(TMTSF)_2ClO_4$ in the $\hat{b}$ and $\hat{c}$ directions. Their explanation agrees quite well with the data and some of the results of their simulations are plotted in Figure 19

It is possible to probe the bandwidth of $(TMTSF)_2ClO_4$ in the $\hat{b}$ and $\hat{c}$ directions because the shape of the resonances in the semiclassical theory is a calculable function of these bandwidths and is sensitive to both of their magnitudes. The shape and therefore the test is most sensitive to the value of t_b, but also somewhat to the value of t_c. Importantly, however, the test is *entirely dependent on the existence of coherent motion out of the ab plane.* Without such coherence there is no notion of a quasi-classical velocity or momentum transverse to the *ab* planes and the averaging effect simply should not exist. The natural prediction for $(TMTSF)_2PF_6$ of the incoherence theory is that these same resonances should be present for that material if and only if the component of field out of the *ac* plane is not sufficiently large for complete incoherence to have set in. If it is large enough, then the behavior should cross over to a broad background with a magnetoresistance depending only on $\vec{B}\cdot\hat{n}_{ab}$ (in fact

the background ΔR_{zz} should be $\sim (\vec{B} \cdot \hat{n}_{ab})^x$ where $x \sim 3/2$), and the background should have dips superimposed where the component of the field perpendicular to the $\hat{a}$ direction points along a real space lattice direction. This occurs because, for these fields, hops along the field have no associated Δk_a and therefore no significant inelasticity; these field are not effective in inducing incoherence. A magnetoresistance with almost exactly this structure is seen [33].

For fields with no component out of the *ac* plane, there are features in the PF_6 magnetoresistance data which are analogous to those seen in ClO_4. The features for PF_6 are much weaker and there is a large background magnetoresistance compared to ClO_4 for fields tilted away from the $\hat{a}$ direction (see Figure 20)

In fact, the identification of the features in the PF_6 magnetoresistance plot with those present for ClO_4 is not in itself compelling, however, the similarity is much clearer if the second derivative with respect to angle is plotted instead [33] as in Figure 21, which clearly demonstrates that the same resonances are present in PF_6 as in ClO_4.

As the field is tilted out of the *ac* plane we expect the *c*-axis hopping to become incoherent and the resonance features should vanish. As the data of Danner *et al.* show, the resonances are gone or nearly gone already for fields a field with a 0.2 Tesla component out of the *ac* plane As the resonances disappear the background magnetoresistance crosses over to the $(\vec{H} \cdot \hat{n}_{ab})^{\frac{3}{2}}$ for fields with projections onto the *bc* plane which are away from magic angle directions; meanwhile the dips at Lebed magic angles appear as the component of the field out of the *ac* plane grows, becoming clear when the field strength out of the *ac* plane reaches 1 Tesla. These effects are as shown in Figures 22 and 23.

Notice that the quasiclassical explanation of [83] can be extended to include fields out of the *ac* plane as shown in Figure 24.

The results of the experiment are in exact agreement with the predictions of the incoherence theory with the exception of the large background present in PF_6 for fields in or nearly in the *ac* plane. The background crosses over to the two dimensional magnetoresistance of the incoherent state smoothly and is clearly a precursor of that resistance, however that resistance is itself not well understood at the present time. Since it is a property of the isolated planes of the incoherent state, which must have a non-Fermi liquid character for the theory to make sense (and of course to have any chance of accounting for the magnetoresistance itself), we need to understand this non-Fermi liquid state well enough to study the effects of magnetic field on it, which has not been possible to this time.

We believe that the degree to which the incoherence theory predicts and explains the experimental data is compelling. Moreover, those features of the data which are as yet inexplicable are in no way in conflict with the theory, whereas they are in fundamental conflict with a three dimensionally coherent picture for $(TMTSF)_2PF_6$. Less exotic sources of incoherence than our proposal, e.g. disorder and finite temperature effects, are also effectively ruled out since the relevant experimental anomalies are clearest at the lowest temperatures and in the cleanest samples; we believe them to be the result of an interaction induced confinement of coherence even in the pure system, zero temperature limit.

Parenthetically, we should remark that the experimental data on $(TMTSF)_2PF_6$ place some very interesting constraints on the breakdown of Fermi liquid theory occurring in the incoherent phase. First, further investigations [84] of the magnetoresistance have revealed

that the 3/2 power law seen in the R_{zz} magnetoresistance is *nonuniversal* and strongly pressure dependent. The non-Fermi liquid state underlying this behavior must therefore support non-universal power laws, a property reminiscent of the Luttinger liquid ideas advocated by Anderson [4]. Second, at the temperatures where the non-Fermi liquid behavior occurs, the conduction band of $(TMTSF)_2PF_6$ is half-filled as the result of a dimerization transition. A successful theory must therefore be consistent with metallic behavior at half-filling in the non-Fermi liquid state. It is also interesting to note that the field required to destroy the coherence and stabilize the non-Fermi liquid state is smaller than the critical field in the b direction. The superconductivity in this material is therefore exotic in the sense that the superconducting state can be entered directly from a state which is clearly non-Fermi liquid. It seems likely that the non-Fermi liquid physics is central to the occurrence of superconductivity at reasonably high temperature in $(TMTSF)_2PF_6$, when all other measured properties are most consistent with repulsive interactions.

In summary, when all of the data on $(TMTSF)_2PF_6$ are taken together, we regard them as providing an experimental demonstration of the existence of an alternative finite $t_\perp$ fixed point at which there is no three dimensional coherence even though $t_\perp$ is not irrelevant. These are exactly the essential, novel features of our proposal and we regard the experiments as demonstrating the existence of a phase of confined coherence, although the question of the precise connection to the Luttinger calculation we have done is still open.

IX. CONCLUSION

Our essential claims in this paper are that there are sound theoretical reasons for believing in the existence of a previously undiscovered state of matter in which transport in one direction has been rendered incoherent by interaction effects, and that, most importantly, there exist experiments which demonstrate the existence of this phase beyond reasonable doubt.

Our initial reason for believing that such a state might exist was the irreconcilability, discussed in the Introduction and in Sec. VII, of the experimental data on the high temperature superconductors with the various existing theoretical proposals. This led us to propose as a natural alternative to previous theoretical suggestions a state in which three dimensional coherence was lost through interactions. We believe that this is a separate question from the renormalization group relevance of $t_\perp$ and that, while the two effects are related, incoherence in general sets in before irrelevance.

Our proposal is based on a close analogy between the incoherence found in the two level system problem (discussed in section II) and single particle hopping between Luttinger liquids (discussed in Sec. III and IV). This analogy suggests that the oscillation frequency of the number difference between two coupled Luttinger liquids can be identified with the magnitude of the coherent, single particle hopping between the liquids. If this oscillation frequency vanishes there is no coherent hopping [85]. In Sec. V we have discussed in detail how the dynamical calculations of Sec. II, III and IV can thus be reliably interpreted to reveal the nature (i.e. coherence or incoherence) of the $t_\perp \neq 0$ groundstate.

For the special case of coupled Luttinger liquids which we have considered in detail theoretically, this leads us to propose a schematic renormalization group diagram for the

two and many chain problems very different from previous ideas [35].

Our picture differs from that reached by other studies of the same problem in that the possibility of an incoherent fixed point for intermediate interactions is included. In general, other works postulated that the system either crosses over to a two dimensional Fermi liquid or has a renormalization group irrelevant $t_\perp$, resulting in a trivial confinement of the electrons to their original chains. In our case, there exists an additional intervening phase which we refer to as a phase of "confined coherence" since, in this phase, coherent transport (though not the electrons) is confined to the chains.

The change from coherent transport in all directions to confined coherence should be thought of as a true transition as discussed in Sec. V. The natural order parameter is the shape of the Fermi surface, specifically the $k_\perp$ dependence of the Fermi surface, since the phase with confined coherence has no higher dimensional Fermi surface and hence no dependence at all on $k_\perp$, while the fully coherent phase has a Fermi surface with warping categorized by some renormalized, coherent $t_\perp$ which is non-zero.

Having introduced this transition, we are able to explain the apparent contradictions in the experimental data on the cuprate superconductors which originally motivated our work (see Sec. VII and [8]). This strongly supports our theoretical proposal, however, the clearest experimental evidence in favor of the existence of the new phase comes from experiments on $(TMTSF)_2PF_6$, where magnetoresistance data demonstrate that the material can undergo a transition from a three dimensional Fermi liquid to a state with no coherent three dimensional transport [33]. This demonstrates a state in which the coherent single particle hopping is indeed vanishing, yet in this state conductivity in the perpendicular direction is not insulating. This rules out the renormalization group irrelevance of $t_\perp$ and thus requires a state of exactly the sort we are proposing. The material is also categorized by an extremely long mean free path, while the relevant experiments are conducted at 0.5 K, smaller than any of the relevant energy scales in the problem. Further, the effects of interest are increasingly pronounced with cleaner samples and lower temperatures, so that neither disorder effects nor finite temperature can explain the results. This was discussed in detail in Sec. VIII along with various other experimental results on $(TMTSF)_2PF_6$ that are uniquely understandable in our theory.

In summary, we have made a theoretical proposal for a new state in order to reconcile features of the experimental data on the high T_c cuprates. Our best calculation suggests that the proposal is viable for sufficiently strongly correlated, anisotropic systems and there exists experiments on the material $(TMTSF)_2PF_6$ that are uniquely understandable in terms of a state in which exactly the proposed incoherence obtains.

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X. APPENDIX

A. Interliquid Hopping Rate

To $O(t_\perp^2)$,

$$P(t) \equiv | \langle O_1 O_2 \mid e^{iH_0 t} e^{-iHt} \mid O_1 O_2 \rangle |^2 \tag{137}$$

is given by

$$1 - P(t) = 2t_\perp^2 L \mathrm{Re} \int_0^t dt_1 \int_0^{t_1} dt_2 \int dx \left\{ \langle c^{(1)}(x,t_1) c^{(1)\dagger}(0,t_2) \rangle \langle c^{(2)\dagger}(x,t_1) c^{(2)}(0,t_2) \rangle + (1 \leftrightarrow 2) \right\} \tag{138}$$

where the superscripts on the electron operators label the liquid in which the operator acts. In obvious notation, the interliquid hopping rates Γ_{ij} defined by

$$\Gamma(t) = \Gamma_{12}(t) + \Gamma_{12}(t) \equiv -\frac{dP(t)}{dt} \tag{139}$$

are then given by

$$\begin{aligned} \Gamma_{12}(t) &= 2t_\perp^2 L \mathrm{Re} \int_0^t dt' \int \frac{dk}{2\pi} \langle c^{(1)}(k,t') c^{(1)\dagger}(k,0) \rangle \langle c^{(2)\dagger}(k,t') c^{(2)}(k,0) \rangle \\ &= 2t_\perp^2 L \mathrm{Re} \int_{-\infty}^{\infty} dt' \, \theta(t') \, \theta(t-t') \int \frac{dk}{2\pi} \mathcal{J}_1^{(1)}(k,t') \mathcal{J}_2^{(2)}(k,-t') \end{aligned} \tag{140}$$

and similarly for Γ_{21}. Here we have introduced the Green's functions

$$\begin{aligned} \mathcal{J}_1(k,t') &\equiv \langle c(k,t') c^\dagger(k,0) \rangle \\ \mathcal{J}_2(k,t') &\equiv \langle c^\dagger(k,0) c(k,t') \rangle \end{aligned}$$

Upon Fourier transforming we obtain

$$\Gamma_{12}(t) = 2t_\perp^2 L \int \frac{d\omega}{2\pi} \frac{\sin \omega t}{\omega} \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} \mathcal{J}_1^{(1)}(k,\omega') \mathcal{J}_2^{(2)}(k,\omega'-\omega) \tag{141}$$

which, in the notation introduced in the text, may be written

$$\Gamma_{12}(t) = 2t_\perp^2 L \int \frac{d\omega}{2\pi} \frac{\sin \omega t}{\omega} A_{12}(\omega) \tag{142}$$

where

$$A_{ij}(\omega) = \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} \mathcal{J}_1^{(i)}(k,\omega') \mathcal{J}_2^{(j)}(k,\omega'-\omega) \tag{143}$$

Note that $\mathcal{J}_{1,2}$ can be expressed in terms of the electron spectral function via

$$\mathcal{J}_{1,2}(k,\omega') = \frac{\rho(k,\omega'-\mu)}{1 + e^{\mp\beta(\omega'-\mu)}} \tag{144}$$

which reduces in the zero temperature limit to

$$\mathcal{J}_{1,2}(k,\omega') = \theta_\pm(\omega'-\mu) \rho(k,\omega'-\mu) \tag{145}$$

B. Coupled Fermi Liquids

At low enough energies, the Fermi liquid spectral function may be taken to be

$$\rho(k,\omega) = \frac{2Z^2\gamma\omega^2}{(\omega - E_k)^2 + Z^2\gamma^2\omega^4}$$

We present the calculation for $A_{12}(\omega)$, which is the interliquid hopping spectral function with potential coherence. We have

$$A_{12}(\omega) = \theta_+(\omega + \mu_2 - \mu_1)\int_{\mu_1}^{\mu_2+\omega}\frac{d\omega'}{2\pi}\int\frac{dk}{2\pi}\rho(k,\omega'-\mu_1)\rho(k,\omega'-\omega-\mu_2) \tag{146}$$

where we have taken $\Delta\mu \equiv \mu_2 - \mu_1 > 0$. For sufficiently small ω, one can approximate one of the ρ's by a δ-function to obtain

$$\begin{aligned}
A_{12}(\omega) &= \theta_+(\omega+\mu_2-\mu_1)\int_0^{\omega+\Delta\mu}\frac{d\omega'}{2\pi}\int\frac{dk}{2\pi}\rho(k,\omega')\rho(k,\omega'-\omega-\Delta\mu) \\
&\sim \theta_+(\omega+\mu_2-\mu_1)\int_0^{\omega+\Delta\mu}\frac{d\omega'}{2\pi}\int\frac{dk}{2\pi}2\pi Z\,\delta(\omega'-vk+\mu_1)\rho(k,\omega'-\omega-\Delta\mu) \\
&= \theta_+(\omega+\Delta\mu)\frac{1}{v}\int_0^{\omega+\Delta\mu}\frac{d\omega'}{2\pi}Z\,\rho((\omega'+\mu_1)/v,\omega'-\omega-\Delta\mu) \\
&= \theta_+(\omega+\Delta\mu)\frac{1}{v}\int_0^{\omega+\Delta\mu}\frac{d\omega'}{2\pi}\frac{2Z^3\gamma(\omega'-\omega-\Delta\mu)^2}{\omega^2+Z^2\gamma^2(\omega'-\omega-\Delta\mu)^4}
\end{aligned} \tag{147}$$

Shifting the integration variable gives

$$\begin{aligned}
A_{12}(\omega) &\sim \theta_+(\omega+\Delta\mu)\frac{1}{v}\int_{-\Delta\mu}^{\omega}\frac{d\omega'}{2\pi}\frac{2Z^3\gamma(\omega'-\omega)^2}{\omega^2+Z^2\gamma^2(\omega'-\omega)^4} \\
&= \theta_+(\omega+\Delta\mu)\frac{1}{v}\left\{\int_0^{\omega} + \int_{-\Delta\mu}^{0}\right\}
\end{aligned} \tag{148}$$

For low energies, $\omega^2 \gg Z^2\gamma^2\omega^4$, the first integral evaluates to $(3\pi v)^{-1}Z^3\gamma\,\omega$. The second integral needs care to evaluate, since it is formally divergent at $\omega = 0$. It can be rewritten as

$$\begin{aligned}
&Z^2\int_0^{\Delta\mu}\frac{d\omega'}{2\pi}\frac{2Z\gamma(\omega'+\omega)^2}{\omega^2+Z^2\gamma^2(\omega'+\omega)^4} \\
&= Z^2\Delta\mu\lim_{\Omega\to 0}\frac{Z\gamma}{\pi}\frac{(\omega+\Omega)^2}{\omega^2+Z^2\gamma^2(\omega+\Omega)^4} \\
&= Z^2\Delta\mu\delta(\omega)
\end{aligned} \tag{149}$$

in the limit of small $\Delta\mu$.

We therefore find

$$A_{12}(\omega) \sim \theta_+(\omega+\Delta\mu)\frac{1}{v}\left\{Z^2\Delta\mu\delta(\omega) + \frac{Z^3\gamma}{3\pi}\omega\right\} \tag{150}$$

the $\delta(\omega)$ piece representing a coherent term, while the term linear in ω is marginal.

C. Coupled Luttinger Liquids

We now consider the case of coupled Luttinger liquids. In order to proceed, we need Luttinger liquid spectral functions. A simple method of obtaining these is presented below.

1. Calculation of Luttinger Liquid Spectral Functions

Fracton Convolution Formula

The key to using spectral function methods for Luttinger liquids is to write the Luttinger liquid electron spectral function as a convolution of "fracton" spectral functions, the latter being sharp δ-functions. The fractons have a precise relation to the charge and spin degrees of freedom and therefore represent in a certain well-defined sense "holons" and "spinons".

We begin with some quite general observations. The method of bosonization allows one to express fermionic fields as exponentials of (essentially free) bosonic fields. For definiteness, suppose one may write the electron destruction operator as

$$\psi(x,t) = \lambda e^{i\phi_a(x,t)} e^{i\phi_b(x,t)} \tag{151}$$

where ϕ_a, ϕ_b are bosonic fields chosen such that the electron Hamiltonian may be rewritten to be harmonic in these fields, ϕ_a and ϕ_b being decoupled at the Hamiltonian level. λ is a (real) parameter chosen to get the dimensions of correlation functions correct. We refer to the operators $e^{i\phi_a}$ and $e^{i\phi_b}$ as "fractons", the name being suggestive of the fact that a real electron is made by "glueing" fractons together via multiplication at the same space-time point.

We now write

$$\begin{aligned}\mathcal{J}_1^{(e)}(k,\omega) &= \frac{\lambda^2}{Z}\sum_{n,m}\int_{-\infty}^{\infty} dt e^{i\omega t}\int dx e^{-ikx}\langle n|\psi(x)|m\rangle\langle m|\psi^\dagger(0)|n\rangle e^{-i(E_m-E_n)t}e^{-\beta E_n} \\ &= \frac{2\pi\lambda^2}{Z}\sum_{n,m} e^{-\beta E_n}\int dx e^{-ikx}\langle n|\psi(x)|m\rangle\langle m|\psi^\dagger(0)|n\rangle\delta(E_n - E_m + \omega)\end{aligned} \tag{152}$$

Since ϕ_a and ϕ_b are independent, all exact eigenstates $|m\rangle$ are of the form $|m_a, m_b\rangle$. Moreover, translation invariance allows one to choose $|m_{a,b}\rangle$ to have definite momentum $k_{a,b}$, so that (using $Z = Z_a Z_b$)

$$\begin{aligned}\mathcal{J}_1^{(e)}(k,\omega) = \lambda^2 &\left\{\frac{1}{Z_a}\sum_{n_a,m_a} e^{-\beta E_{n_a}}\left|\langle n_a|e^{i\phi_a(0)}|m_a\rangle\right|^2\right\}\left\{\frac{1}{Z_b}\sum_{n_b,m_b} e^{-\beta E_{n_b}}\left|\langle n_b|e^{i\phi_b(0)}|m_b\rangle\right|^2\right\} \\ &2\pi\delta(\omega + E_{n_a} + E_{n_b} - E_{m_a} - E_{m_b})\; 2\pi\delta(k - k_{m_a} - k_{m_b})\end{aligned} \tag{153}$$

But

$$\begin{aligned}\mathcal{J}_1^{(a)}(k,\omega) &\equiv \int_{-\infty}^{\infty} dt e^{i\omega t}\int dx e^{-ikx}\langle e^{i\phi_a(x,t)}e^{-i\phi_a(0,0)}\rangle \\ &= \frac{1}{Z_a}\sum_{n_a,m_a} e^{-\beta E_{n_a}}\left|\langle n_a|e^{i\phi_a(0)}|m_a\rangle\right|^2 2\pi\delta(k - k_{m_a})\; 2\pi\delta(\omega + E_{n_a} - E_{m_a})\end{aligned} \tag{154}$$

from which we obtain the convolution formula

$$\mathcal{J}_1^{(e)}(k,\omega) = \frac{\lambda^2}{(2\pi)^2}\int dk_1 \int dk_2 \int d\omega_1 \int d\omega_2 \\ \mathcal{J}_1^{(a)}(k_1,\omega_1)\mathcal{J}_1^{(b)}(k_2,\omega_2)\ \delta(\omega-\omega_1-\omega_2)\ \delta(k-k_1-k_2) \tag{155}$$

The generalization to three or more fracton fields is straightforward and in fact necessary for dealing with spin-charge separated Luttinger liquids.

Fracton Spectral Functions

It remains to give expressions for the fracton spectral functions, from which the electron spectral function may be constructed using the above convolution formula. The space-time correlation function for a fracton of weight p is

$$\langle e^{i\phi(x,t)}e^{i\phi(0,0)}\rangle \sim \left(\frac{i\pi a}{v\beta}\right)^p \left\{\sinh\left[\frac{\pi(x+ia-vt)}{v\beta}\right]\right\}^{-p} \tag{156}$$

from which one obtains

$$\mathcal{J}_1^{(p)}(k,\omega) = \left(\frac{i\pi a}{v\beta}\right)^p\ 2\pi\delta(\omega-vk)\int_{-\infty}^{\infty}\frac{dz\ e^{-ikz}}{\left[\sinh\left(\frac{\pi(z+ia)}{v\beta}\right)\right]^p} \tag{157}$$

for the fracton spectral function. Here, β is inverse temperature, v the velocity of the relevant bosonic excitations, ϕ, and a is a short-distance cutoff.

For our purposes, we shall only need the zero-temperature limit:

$$\mathcal{J}_1^{(p)}(k,\omega) \overset{\beta\to\infty}{\sim} \left(\frac{i\pi a}{v\beta}\right)^p\ 2\pi\delta(\omega-vk)\int_{-\infty}^{\infty}\frac{dz\ e^{-ikz}(v\beta)^p}{[\pi(z+ia)]^p} \\ = \ (ia)^p\ 2\pi\delta(\omega-vk)\int_{-\infty}^{\infty}\frac{dz\ e^{-ikz}}{(z+ia)^p} \tag{158}$$

Choosing the cut for z^{-p} to be along the positive real axis, we may deform the contour of integration to obtain

$$\int_{-\infty}^{\infty}\frac{dz\ e^{-ikz}}{(z+ia)^p} = (1-e^{-i2\pi p})\ \theta_+(k)\,k^{p-1}\int_0^{\infty}d\theta\ e^{-i\theta}\theta^{-p} \tag{159}$$

The θ-integral may be performed by deforming the contour to run up the negative imaginary axis. A change of variable to $x = i\theta$ gives

$$\mathcal{J}_1^{(p)}(k,\omega) = a^p 2\pi\delta(\omega-vk)(1-e^{-i2\pi p})\ \theta_+(k)\,k^{p-1}\ e^{i\pi p}(-i)\int_0^{\infty}dx\ e^{-x}x^{-p} \\ = 4\pi\ a^p\sin(\pi p)\ \Gamma(1-p)\ k^{p-1}\theta_+(\omega)\,\delta(\omega-vk) \\ = \frac{4\pi^2}{\Gamma(p)}a^p\ k^{p-1}\theta_+(\omega)\delta(\omega-vk) \tag{160}$$

the final line following from the identity $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$.

Spinless Luttinger liquid

From the previous section, we may write

$$\mathcal{J}_1(k,\omega) \propto \int dk_1 dk_2 \int d\omega_1 d\omega_2 \delta(\omega - \mu - \omega_1 - \omega_2)\delta(k - k_F - k_1 - k_2)$$
$$\theta_+(\omega_1)(\omega_1/v_c)^p \delta(\omega_1 - v_c k_1)\theta_+(\omega_2)(\omega_2/v_c)^{p-1}\delta(\omega_2 + v_c k_2) \tag{161}$$

A similar expression for $\mathcal{J}_2$ holds. Then

$$\begin{aligned} A_{12}(\omega) &\propto \int \frac{d\omega'}{2\pi}\int \frac{dk}{2\pi}\int_0^\infty d\omega_1 d\omega_2 \delta(\omega' - \mu_1 - \omega_1 - \omega_2)\delta(k - k_F^{(1)} - \frac{\omega_1}{v_c} + \frac{\omega_2}{v_c})(\omega_1/v_c)^p(\omega_2/v_c)^{p-1} \\ &\int_{-\infty}^0 d\omega_1' d\omega_2' \delta(\omega' - \omega - \mu_2 - \omega_1' - \omega_2')\delta(k - k_F^{(2)} - \frac{\omega_1'}{v_c} + \frac{\omega_2'}{v_c})(\omega_1'/v_c)^p(\omega_2'/v_c)^{p-1} \\ &= v_c^{3-4p}\int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2' \delta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - (\omega + \Delta\mu)) \\ &\delta(\omega_1 + \omega_1' - \omega_2 - \omega_2' - v_c\Delta k)\omega_1^p \omega_2^{p-1}(\omega_1')^p(\omega_2')^{p-1} \end{aligned} \tag{162}$$

To get some idea of how to evaluate these integrals, let us first consider the case $\Delta\mu = 0$:

$$A_{12}(\omega)_{\Delta\mu=0} \propto \int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2' \delta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - \omega)$$
$$\delta(\omega_1 + \omega_1' - \omega_2 - \omega_2')\omega_1^p \omega_2^{p-1}(\omega_1')^p(\omega_2')^{p-1} \tag{163}$$

Rescaling of the variables of integration, $\omega_i = \omega x_i$ etc., immediately yields $A_{12}(\omega)_{\Delta\mu=0} \propto \omega^{4p}$. However, it is instructive to be more explicit:

$$\begin{aligned} A_{12}(\omega)_{\Delta\mu=0} &\propto \int_0^\infty d\omega_2 d\omega_1' d\omega_2' \delta(2(\omega_2 + \omega_2') - \omega)(\omega_2 + \omega_2' - \omega_1')^p \theta_+(\omega_2 + \omega_2' - \omega_1')\omega_2^{p-1}(\omega_1')^p(\omega_2')^{p-1} \\ &= \int_0^\infty d\omega_1' d\omega_2' (\omega/2 - \omega_1')^p(\omega/2 - \omega_2')^{p-1}(\omega_1')^p(\omega_2')^{p-1}\theta_+(\omega/2 - \omega_1')\theta_+(\omega/2 - \omega_2') \\ &= \theta_+(\omega)\left\{\int_0^{\omega/2} d\omega_1'(\omega/2 - \omega_1')^p(\omega_1')^p\right\}\left\{\int_0^{\omega/2} d\omega_2'(\omega/2 - \omega_2')^{p-1}(\omega_2')^{p-1}\right\} \\ &\propto \theta_+(\omega)\omega^{2p+1}\omega^{2p-1} = \theta_+(\omega)\omega^{4p} \end{aligned} \tag{164}$$

For $\Delta\mu > 0$ write (defining $\Delta \equiv \Delta\mu - v_c\Delta k \equiv (v - v_c)\Delta k = (1 - v/v_c)\Delta k$)

$$\begin{aligned} A_{12}(\omega) &\propto \int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2' \delta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - (\omega + \Delta\mu)) \\ &\delta(\omega_1 + \omega_1' - \omega_2 - \omega_2' - v_c\Delta k)\omega_1^p\omega_2^{p-1}(\omega_1')^p(\omega_2')^{p-1} \\ &= \int_{-\Delta\mu}^\infty d\omega_1 \int_0^\infty d\omega_2 d\omega_1' d\omega_2' \delta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - \omega) \\ &\delta(\omega_1 + \omega_1' - \omega_2 - \omega_2' + \Delta)(\omega_1 + \Delta\mu)^p\omega_2^{p-1}(\omega_1')^p(\omega_2')^{p-1} \\ &= \int_{-\Delta\mu}^\infty d\omega_1 \int_0^\infty d\omega_2 d\omega_2' \delta(2(\omega_2 + \omega_2') - (\omega + \Delta))(\omega_1 + \Delta\mu)^p\omega_2^{p-1}(\omega_2')^{p-1} \\ &(\omega_2 + \omega_2' - \omega_1 - \Delta)^p\theta_+(\omega_2 + \omega_2' - \omega_1 - \Delta) \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\Delta\mu}^{\infty} d\omega_1 \int_0^{\infty} d\omega_2' (\omega_1 + \Delta\mu)^p (\omega_2')^{p-1} \\
&\left(\frac{(\omega+\Delta)}{2} - \omega_2'\right)^{p-1} \theta_+ \left(\frac{(\omega+\Delta)}{2} - \omega_2'\right) \left(\frac{(\omega-\Delta)}{2} - \omega_1\right)^{p} \theta_+ \left(\frac{(\omega-\Delta)}{2} - \omega_1\right) \\
&= \left\{ \int_0^{\infty} d\omega_1 \omega_1^p \left(\frac{(\omega-\Delta)}{2} + \Delta\mu - \omega_1\right)^{p} \theta_+ \left(\frac{(\omega-\Delta)}{2} + \Delta\mu - \omega_1\right) \right\} \\
&\left\{ \int_0^{\infty} d\omega_2' (\omega_2')^{p-1} \left(\frac{(\omega+\Delta)}{2} - \omega_2'\right)^{p-1} \theta_+ \left(\frac{(\omega+\Delta)}{2} - \omega_2'\right) \right\} \qquad (165)
\end{aligned}
$$

Using $\Delta = (1 - v/v_c)\Delta k$ this simplifies to

$$
\begin{aligned}
A_{12}(\omega) \propto \theta_+ \left[\omega + \left(1 - \frac{v_c}{v}\right)\Delta\mu\right] \theta_+ \left[\omega + \left(1 + \frac{v_c}{v}\right)\Delta\mu\right] \\
\left(\omega + \left(1 - \frac{v_c}{v}\right)\Delta\mu\right)^{2p-1} \left(\omega + \left(1 + \frac{v_c}{v}\right)\Delta\mu\right)^{2p+1} \qquad (166)
\end{aligned}
$$

It is straightforward to show that $A_{21}(\Delta\mu, \omega) = A_{12}(-\Delta\mu, \omega)$. Without loss of generality, we may take $v - v_c < 0$, so that

$$
A_{12}(\omega) \propto \theta_+ \left[\omega - \left(\frac{v_c}{v} - 1\right)\Delta\mu\right] \left(\omega - \left(\frac{v_c}{v} - 1\right)\Delta\mu\right)^{2p-1} \left(\omega + \left(\frac{v_c}{v} + 1\right)\Delta\mu\right)^{2p+1} \qquad (167)
$$

and

$$
A_{21}(\omega) \propto \theta_+ \left[\omega - \left(\frac{v_c}{v} + 1\right)\Delta\mu\right] \left(\omega + \left(\frac{v_c}{v} - 1\right)\Delta\mu\right)^{2p-1} \left(\omega + \left(\frac{v_c}{v} + 1\right)\Delta\mu\right)^{2p+1} \qquad (168)
$$

Chiral Luttinger liquid

Here we have charge- and spin-excitations with different velocities, but no anomalous exponent. The prescription for glueing spin- and charge-fractons together to give the electron spectral function is given by

$$
\begin{aligned}
\mathcal{J}_{(1,2)}(k,\omega) = \frac{1}{(2\pi)^2} \int dk_1 \int dk_2 \int d\omega_1 \int d\omega_2 \mathcal{J}^{(c)}_{(1,2)}(k_1,\omega_1) \mathcal{J}^{(s)}_{(1,2)}(k_2,\omega_2) \\
\delta(\omega - \mu - \omega_1 - \omega_2)\delta(k - k_F - k_1 - k_2) \qquad (169)
\end{aligned}
$$

where

$$
\mathcal{J}_1^{(c)}(k,\omega) \propto \theta_+(\omega)\delta(\omega - v_c k) \left(\frac{\omega}{v_c}\right)^{-1/2} \qquad (170)
$$

$$
\mathcal{J}_1^{(s)}(k,\omega) \propto \theta_+(\omega)\delta(\omega - v_s k) \left(\frac{\omega}{v_s}\right)^{-1/2} \qquad (171)
$$

$$
\mathcal{J}_2^{(c)}(k,\omega) \propto \theta_-(\omega)\delta(\omega - v_c k) \left(\frac{-\omega}{v_c}\right)^{-1/2} \qquad (172)
$$

$$
\mathcal{J}_2^{(s)}(k,\omega) \propto \theta_-(\omega)\delta(\omega - v_s k) \left(\frac{-\omega}{v_s}\right)^{-1/2} \qquad (173)
$$

It is simple to show that if $\Delta\mu = 0$ then $A_{12}(\omega) = A_{21}(\omega) = 0$, so we begin with the general case of $\Delta\mu > 0$:

$$\begin{aligned}
A_{12}(\omega) &= \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} \mathcal{J}_1^{(1)}(k,\omega') \mathcal{J}_2^{(2)}(k,\omega'-\omega) \\
&\propto \int \frac{d\omega'}{2\pi} \int \frac{dk}{2\pi} \\
&\quad \int_0^\infty d\omega_1 d\omega_2 \delta(\omega' - \mu_1 - \omega_1 - \omega_2) \delta(k - k_F^{(1)} - \frac{\omega_1}{v_c} - \frac{\omega_2}{v_s}) (\omega_1/v_c)^{-1/2} (\omega_2/v_s)^{-1/2} \\
&\quad \int_0^\infty d\omega_1' d\omega_2' \delta(\omega' - \omega - \mu_2 + \omega_1' + \omega_2') \delta(k - k_F^{(2)} + \frac{\omega_1'}{v_c} + \frac{\omega_2'}{v_s}) (\omega_1'/v_c)^{-1/2} (\omega_2'/v_s)^{-1/2} \\
&\propto \int_0^\infty d\omega_1 d\omega_2 d\omega_1' d\omega_2' \delta(\omega_1 + \omega_2 + \omega_1' + \omega_2' - (\omega + \Delta\mu)) \\
&\quad \delta\left(\frac{(\omega_1 + \omega_1')}{v_c} + \frac{(\omega_2 + \omega_2')}{v_s} - \Delta k \right) (\omega_1/v_c)^{-1/2} (\omega_2/v_s)^{-1/2} (\omega_1'/v_c)^{-1/2} (\omega_2'/v_s)^{-1/2} \\
&\propto \int_0^\infty d\omega_2 d\omega_1' d\omega_2' \delta(-\Delta - (\omega_2 + \omega_2')\Delta v/v_s - \omega) (\omega_2/v_s)^{-1/2} (\omega_1'/v_c)^{-1/2} (\omega_2'/v_s)^{-1/2} \\
&\quad \left(\Delta k - \frac{(\omega_2 + \omega_2')}{v_s} - \frac{\omega_1'}{v_c} \right)^{-1/2} \theta_+ \left(\Delta k - \frac{(\omega_2 + \omega_2')}{v_s} - \frac{\omega_1'}{v_c} \right) \\
&\propto \frac{v_s}{\Delta v} \left\{ \int_0^\infty d\omega_1' (\omega_1'/v_c)^{-1/2} \left(\Delta k + \frac{(\omega + \Delta)}{\Delta v} - \frac{\omega_1'}{v_c} \right)^{-1/2} \theta_+ \left(\Delta k + \frac{(\omega + \Delta)}{\Delta v} - \frac{\omega_1'}{v_c} \right) \right\} \\
&\quad \left\{ \int_0^\infty d\omega_2' (\omega_2'/v_s)^{-1/2} \left(-\frac{(\omega + \Delta)}{\Delta v} - \frac{\omega_2'}{v_s} \right)^{-1/2} \theta_+ \left(-\frac{(\omega + \Delta)}{\Delta v} - \frac{\omega_2'}{v_s} \right) \right\} \qquad (174)
\end{aligned}$$

Using $\int_0^x dt\, t^{-1/2}(x-t)^{-1/2} = \int_0^1 dt\, t^{-1/2}(1-t)^{-1/2}$ (i.e. independent of x), we obtain

$$A_{12}(\omega) \propto \frac{1}{\Delta v} \theta_+(v_c \Delta k - \Delta\mu - \omega) \theta_+(\omega + \Delta\mu - v_s \Delta k) \qquad (175)$$

$$A_{21}(\omega) = 0 \qquad (176)$$

that is, $A_{12}(\omega)$ has constant non-vanishing weight in the interval $\omega \in [v_s\Delta k - \Delta\mu, v_c\Delta k - \Delta\mu]$. Note that, in the limit $\Delta v \to 0$, this step function of width Δv, height $\propto 1/\Delta v$, goes over to a δ-function, as it should, for the limit $\Delta v \to 0$ is the limit of free electrons.

Spinny Luttinger liquid

This case has aspects of both the spinless Luttinger liquid, and the chiral Luttinger liquid. We shall see that the high frequency behavior is essentially the same as that of the spinless case, while the effect of spin-charge separation is to destroy the low frequency divergence in $A_{12}(\omega)$ which was present in the spinless case.

For the electron spectral function we have the convolution

$$\begin{aligned}
\mathcal{J}_1(k,\omega) &\propto \int dk_1 dk_2 dk_3 \int d\omega_1 d\omega_2 d\omega_3 \delta(\omega - \mu - \sum_i \omega_i) \delta(k - k_F - \sum_i k_i) \\
&\quad \delta(\omega_1 - v_c k_1) \theta_+(\omega_1) (\omega_1/v_c)^{\alpha - 1/2} \delta(\omega_2 - v_s k_2) \theta_+(\omega_2) (\omega_2/v_s)^{-1/2} \delta(\omega_3 + v_c k_3) \theta_+(\omega_3) (\omega_3/v_c)^{\alpha - 1} \\
&\propto \int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega - \mu - \sum_i \omega_i) \delta\left(k - k_F - \frac{(\omega_1 - \omega_3)}{v_c} - \frac{\omega_2}{v_s} \right)
\end{aligned}$$

$$(\omega_1/v_c)^{\alpha-1/2}(\omega_2/v_s)^{-1/2}(\omega_3/v_c)^{\alpha-1} \tag{177}$$

Similarly

$$\mathcal{J}_2(k,\omega) \propto \int_0^\infty d\omega_1' d\omega_2' d\omega_3' \delta(\omega-\mu+\sum_i \omega_i')\delta\left(k-k_F+\frac{(\omega_1'-\omega_3')}{v_c}+\frac{\omega_2'}{v_s}\right)$$
$$(\omega_1'/v_c)^{\alpha-1/2}(\omega_2'/v_s)^{-1/2}(\omega_3'/v_c)^{\alpha-1} \tag{178}$$

For $\Delta\mu > 0$ we calculate $A_{12}(\omega)$ as follows:

$$\begin{aligned}
A_{12}(\omega) &\propto \int \frac{d\omega'}{2\pi}\int\frac{dk}{2\pi}\int_0^\infty d\omega_1 d\omega_2 d\omega_3 \delta(\omega'-\sum_i\omega_i)\delta\left(k-k_F-\frac{(\omega_1-\omega_3)}{v_c}-\frac{\omega_2}{v_s}\right)\omega_1^{\alpha-1/2}\omega_2^{-1/2}\omega_3^{\alpha-1} \\
&\quad \int_0^\infty d\omega_1' d\omega_2' d\omega_3' \delta(\omega-\omega'-\Delta\mu+\sum_i\omega_i')\delta\left(k-k_F+\frac{(\omega_1'-\omega_3')}{v_c}+\frac{\omega_2'}{v_s}\right)(\omega_1')^{\alpha-1/2}(\omega_2')^{-1/2}(\omega_3')^{\alpha-1} \\
&\propto \int_{-\Delta\mu}^\infty d\omega_1 \int_0^\infty d\omega_2 d\omega_3 d\omega_1' d\omega_2' d\omega_3' \\
&\quad \delta(\sum_i(\omega_i+\omega_i')-\omega)\delta((\omega_1+\omega_1')-(\omega_3+\omega_3')+(v_c/v_s)(\omega_2+\omega_2')+\Delta) \\
&\quad (\omega_1+\Delta\mu)^{\alpha-1/2}\omega_2^{-1/2}\omega_3^{\alpha-1}(\omega_1')^{\alpha-1/2}(\omega_2')^{-1/2}(\omega_3')^{\alpha-1} \\
&\propto \int_{-\Delta\mu}^\infty d\omega_1 \int_0^\infty d\omega_2 d\omega_3 d\omega_2' d\omega_3' \delta\left(2(\omega_3+\omega_3')+\left(1-\frac{v_c}{v_s}\right)(\omega_2+\omega_2')-(\omega+\Delta)\right) \\
&\quad (\omega_1+\Delta\mu)^{\alpha-1/2}\omega_2^{-1/2}\omega_3^{\alpha-1}(\omega_2')^{-1/2}(\omega_3')^{\alpha-1} \\
&\quad \left((\omega_3+\omega_3')-\frac{v_c}{v_s}(\omega_2+\omega_2')-(\omega_1+\Delta)\right)^{\alpha-1/2}\theta_+\left((\omega_3+\omega_3')-\frac{v_c}{v_s}(\omega_2+\omega_2')-(\omega_1+\Delta)\right) \\
&= \int_0^\infty d\omega_1 d\omega_2 d\omega_2' d\omega_3' \omega_1^{\alpha-1/2}\omega_2^{-1/2}(\omega_2')^{-1/2}(\omega_3')^{\alpha-1} \\
&\quad \left(\frac{1}{2}\left[(\omega+\Delta)-\left(1-\frac{v_c}{v_s}\right)(\omega_2+\omega_2')\right]-\omega_3'\right)^{\alpha-1}\theta_+\left(\frac{1}{2}\left[(\omega_1+\Delta)-\left(1-\frac{v_c}{v_s}\right)(\omega_2+\omega_2')\right]-\omega_3'\right) \\
&\quad \left(\frac{1}{2}\left[(\omega-\Delta)-\left(1+\frac{v_c}{v_s}\right)(\omega_2+\omega_2')+2\Delta\mu\right]-\omega_1\right)^{\alpha-1/2} \\
&\quad \theta_+\left(\frac{1}{2}\left[(\omega-\Delta)-\left(1+\frac{v_c}{v_s}\right)(\omega_2+\omega_2')+2\Delta\mu\right]-\omega_1\right) \\
&\propto \int_0^\infty d\omega_2 d\omega_2' \omega_2^{-1/2}(\omega_2')^{-1/2}\left((\omega+\Delta)-\left(1-\frac{v_c}{v_s}\right)(\omega_2+\omega_2')\right)^{2\alpha-1} \\
&\quad \theta_+\left((\omega+\Delta)-\left(1-\frac{v_c}{v_s}\right)(\omega_2+\omega_2')\right) \\
&\quad \left((\omega-\Delta)-\left(1+\frac{v_c}{v_s}\right)(\omega_2+\omega_2')+2\Delta\mu\right)^{2\alpha}\theta_+\left((\omega-\Delta)-\left(1+\frac{v_c}{v_s}\right)(\omega_2+\omega_2')+2\Delta\mu\right) \\
&= \int_0^\infty d\omega_2 d\omega_2' \omega_2^{-1/2}(\omega_2')^{-1/2}\left((\omega+\Delta)+\frac{\Delta v}{v_s}(\omega_2+\omega_2')\right)^{2\alpha-1}\theta_+\left((\omega+\Delta)+\frac{\Delta v}{v_s}(\omega_2+\omega_2')\right) \\
&\quad \left((\omega-\Delta+2\Delta\mu)-\frac{2\bar{v}}{v_s}(\omega_2+\omega_2')\right)^{2\alpha}\theta_+\left((\omega-\Delta+2\Delta\mu)-\frac{2\bar{v}}{v_s}(\omega_2+\omega_2')\right)
\end{aligned} \tag{179}$$

To proceed further we need to calculate the generic integral

$$I(a,b) = \int_0^\infty dx \int_0^\infty dy x^{-1/2} y^{-1/2} (a-(x+y))^{2\alpha} (b+(x+y))^{2\alpha-1} \theta_+(a-(x+y)) \theta_+(b+(x+y))$$

This double integral can be reduced to a single integral via change of variables

$$\eta = \frac{1}{\sqrt{2}}(x+y)$$
$$\xi = \frac{1}{\sqrt{2}}(x-y)$$

giving

$$\begin{aligned} I(a,b) &= \sqrt{2} \int_0^\infty d\eta \int_{-\eta}^{\eta} d\xi \frac{(a-\sqrt{2}\eta)^{2\alpha}(b+\sqrt{2}\eta)^{2\alpha-1}}{(\eta^2-\xi^2)}^{1/2} \theta_+(a-\sqrt{2}\eta)\theta_+(b+\sqrt{2}\eta) \\ &= \sqrt{2} \int_0^\infty d\eta (a-\sqrt{2}\eta)^{2\alpha}(b+\sqrt{2}\eta)^{2\alpha-1} \theta_+(a-\sqrt{2}\eta)\theta_+(b+\sqrt{2}\eta) \left\{ \int_{-1}^{1} \frac{d(\xi/\eta)}{1-(\xi/\eta)^2)^{1/2}} \right\} \\ &= \pi \int_0^\infty d\eta (a-\eta)^{2\alpha}(b+\eta)^{2\alpha-1} \theta_+(a-\eta)\theta_+(b+\eta) \end{aligned} \tag{180}$$

We may then write

$$A_{12}(\omega) \propto \left(\frac{\bar{v}}{v_s}\right)^{2\alpha} \left(\frac{\Delta v}{v_s}\right)^{2\alpha-1} I(a,b)$$

with

$$a = \frac{v_s}{\bar{v}}(\omega + \Delta\mu + v_c \Delta k)$$
$$b = \frac{v_s}{\Delta v}(\omega + \Delta\mu - v_c \Delta k)$$

We can reduce $I(a,b)$ further. There are two cases (from here on we write $v\Delta k$ for $\Delta\mu$):
(i) $b > 0$ (*i.e.* $\omega > (v_c - v)\Delta k$)

$$\begin{aligned} I(a,b) &= \pi \int_0^a dx\ (a-x)^{2\alpha}(b+x)^{2\alpha-1} \\ &= \frac{\pi}{(1+2\alpha)}\ a^{2\alpha+1} b^{2\alpha-1}\ {}_2F_1\left(1, 1-2\alpha; 2+2\alpha; -\frac{a}{b}\right) \end{aligned} \tag{181}$$

(ii) $b < 0, a+b > 0$ (*i.e.* $(v_s - v)\Delta k < \omega < (v_c - v)\Delta k$)

$$\begin{aligned} I(a,b) &= \pi \int_{-b}^{a} dx\ (a-x)^{2\alpha}(b+x)^{2\alpha-1} \\ &= \pi \int_0^{a+b} dy\ (a+b-y)^{2\alpha} y^{2\alpha-1} \\ &= (a+b)^{4\alpha} \int_0^1 dt\ t^{2\alpha-1}(1-t)^{2\alpha} \\ &= (a+b)^{4\alpha} \frac{\Gamma(2\alpha)\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)} \end{aligned} \tag{182}$$

Upon substituting for a and b, and reinserting the appropriate prefactors, one obtains the results for the interliquid hopping spectral function presented in the main body of the paper.

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occur for Luttinger liquids.
[48] Here we considered a δN at only the right Fermi point, if we add equal numbers at both Fermi points then $E_F^1 - E_F^2 = (k_F^1 - k_F^2)v_N$, which is again in general different from $v_S(k_F^1 - k_F^2) = \sqrt{v_J v_N}(k_F^1 - k_F^2)$ but may be smaller if the interaction is attractive.
[49] If we instead choose the definition of high energy to be energies above those making a significant contribution to $\langle \delta N(t) \rangle$, the result is $O(1)$ for all α so that it appears reasonable that the incoherent hops could completely wipe out the coherence for any α for one extra particle.
[50] This expression is valid only for small α and the effect is *not* divergent as $\alpha \to 1/4$.
[51] We emphasize that this is an interpretation of the results obtained. In general, at least half of the poles obtained in the Green's function occurs at complex frequency and the resulting Green's function is incompatible with a Fermi liquid groundstate. The calculation requires interpretation, and while that of the author, that the splitting of the poles and the renormalization relevance of $t_\perp$ indicate flow to a higher dimensionally coherent state, is reasonable, other possibilities are not excluded.
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[58] While we have not actually proven it, we believe it is extremely reasonable to expect that, if the states connected to the $t_\perp = 0$ groundstates (at fixed N and J) have enegies effectively non-degenerate with those groundstates, then the states connected by to the true $t_\perp \neq 0$ groundstatte by $t_\perp$ should also have this non-degeneracy. This is essentially the assumption made in the TLS problem in [31].
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FIGURES

FIG. 1. Behavior of the poles of the Green's function for $k = k_F$ in the approximation discussed in the text. There is no physically sensible solution for $\alpha > 1/4$ since the poles move into physically inaccessible regions as $\alpha \to 1/4$.

FIG. 2. Behavior of the poles of the Green's function for $k \neq k_F$ in the approximation discussed in the text. There is a physically allowed solution for $t_\perp > 0$ and $\alpha > 1/4$, however, for $\alpha > 1/3$ the alloed pole is shifted to energies with a lower, not higher real part. In addition, the pole for $t_\perp < 0$ does not exist for any $\alpha > 1/4$ or for k (measured from k_F) too large, as discussed in the text. Instead a new pole appears on the real axis with unphysical properties, as discussed in the text.

FIG. 3. Behavior of the poles of the Green's function for $k \neq k_F$ and spin charge separation included in the approximation discussed in the text. There is a physically allowed pole for $t_\perp > 0$ and $\alpha > 1/4$ only for sufficiently small k (measured from k_F) and the pole lies to the left of $v_\rho k$ for $\alpha > 1/6$. The pole for $t_\perp < 0$ which is continuously connected to the pole for $\alpha = 0$ $v_\sigma = v_\rho$ is not shown, but behaves essentially as in the spinless case, while the other pole for $t_\perp < 0$ disperses along the chains like $v_\sigma k$.

FIG. 4. Electron spectral function (right moving part, $\rho_+(q, \omega)$ where $q \equiv k - k_F$) in a spin-charge separated Luttinger liquid (from J. Voit, Ref. [44]). The exponent γ_ρ in the figure is the same as α in our notation.

FIG. 5. The interliquid hopping spectral function for *spinless* Luttinger liquids for various values of α. Here $\omega_l = (v_c - v)\Delta k$ and ω_u is the ultraviolet cutoff of order v/a. The plots do not include the weak power law cutoff dependent prefactor. Note that for $\alpha = 0$, $A_{12}(\omega) \propto \delta(\omega)$.

FIG. 6. The interliquid hopping spectral function for *spinny* Luttinger liquids, for various values of α. Here $\omega_l = (v_s - v)\Delta k$, $\omega_i = (v_c - v)\Delta k$ and ω_u is the ultraviolet cutoff of order v/a. The plots do not include the weak power law cutoff dependent prefactors. The vertical arrow is the $\alpha = 0$ spectral function, $A_{12}(\omega) \propto \delta(\omega)$.

FIG. 7. Temperature dependence of the in-plane (upper panel) and inter-plane (lower panel) resistivity for single crystals of $La_{2-x}Sr_xCuO_4$ with various compositions in the metallic phase (from Ref. [20]).

FIG. 8. In-plane (ρ_{ab}, closed circles) and interlayer (ρ_c, open squares) electrical resistivity of Sr_2RuO_4 plotted against the square of the temperature. The solid lines represent the fits below 25 K: $\rho = \rho_0 + AT^2$ (from Y. Maeno *et al.*, preprint (1995)).

FIG. 9. C-axis optical conductivity spectra below 2.0 eV for $La_{2-x}Sr_xCuO_4$. Expanded spectra in the low-energy region are shown in the inset (from Ref. [22]).

FIG. 10. (a)c-axis reflectivity of $YBa_2Cu_3O_{6+x}$ for several x at room temperature. The inset shows the a-axis reflectivity. (b) Real part of the c-axis optical conductivity of $YBa_2Cu_3O_{6+x}$. The inset shows the a-axis conductivity. (from Ref. [21]).

FIG. 11. The temperature dependence of the c-axis conductivity spectra for $YBa_2Cu_3O_{6+x}$ for various x. (from Ref. [23]).

FIG. 12. Real and imaginary parts of the electronic dynamical conductivity of fully oxygenated $YBa_2Cu_3O_{7-\delta}$ crystals at different temperatures. The electronic conductivity is estimated by subtracting the phononic contribution, which was fitted by five Lorentz oscillators. (from Ref. [69]).

FIG. 13. Resistance along the most conducting direction (in milliohms) as a function of magnetic field strength and orientation. Field was rotated in the bc plane and angle Θ is defined so that ± 90 degrees coincide with the b direction.

FIG. 14. Resistance along the least conducting direction (in ohms) as a function of magnetic orientation. Field strenght was 4 Tesla. The field was rotated in the bc plane and angle Θ is defined so that ± 100 gradians coincide with the b direction.

FIG. 15. Resistance along the most conducting direction (in milliohms) as a function of magnetic field strength and orientation. Field was ramped up for fixed orientation in the bc plane near to magic angles. The angle Θ is defined so that ± 90 degrees coincides with the b direction. As explained in the text, after the destruction of superconductivity, the angle dependence is pronounced in the central dip, but emerges more slowly for the off-center dip.

FIG. 16. Log of the resistance along the most conducting direction (in milliohms) as a function of log of the component of the magnetic field perpendicular to the ab plane. As explained in the text, the data are expected to scale away from the magic angle dips.

FIG. 17. Log of the resistance along the least conducting direction (in ohms) as a function of log (base 10) of the component of the magnetic field (in Tesla) perpendicular to the ab plane. As explained in the text, the data are expected to scale away from the magic angle dips.

FIG. 18. Resonances in conducitivity in the least conducting direction for $(TMTSF)_2ClO_4$ as a function of field orientation. Magnetic fields of various strengths were rotated in the ac plane and the resistance (in ohms) is plotted as a function of angle in the ac plane, measured from $\hat{a}$.

FIG. 19. Calculated resonances in conductivity in the least conducting direction for $(TMTSF)_2ClO_4$ as a function of field orientation. Magnetic fields of various strengths were rotated in the *ac* plane and the resistance (in ohms) is plotted as a function of angle in the *ac* plane, measured from $\hat{a}$.

FIG. 20. Measured resonances in conductivity in the least conducting direction for $(TMTSF)_2PF_6$ as a function of field orientation. Magnetic fields of various strengths were rotated in the *ac* plane and the resistance (in ohms) is plotted as a function of angle in the *ac* plane, measured from $\hat{a}$.

FIG. 21. Comparison of the measured resonances in the second derivative of the conductivity in the least conducting direction for $(TMTSF)_2ClO_4$ and $(TMTSF)_2PF_6$ as a function of field orientation. Angle in the *ac* plane is measured from $\hat{a}$. PF_6 data are offset for clarity.

FIG. 22. Replacement of the Danner resonances in the conductivity in the least conducting direction for $(TMTSF)_2PF_6$ with Lebed magic angle effects. Angle in the *ac* plane is measured from $\hat{a}$.

FIG. 23. Replacement of the Danner resonances in the second derivative of the conductivity in the least conducting direction for $(TMTSF)_2PF_6$ with Lebed magic angle effects. Angle in the *ac* plane is measured from $\hat{a}$.

FIG. 24. Theoretical behavior of the Danner resonances for fields out of the *ac* plane. Materials values are those used for $(TMTSF)_2ClO_4$. Notice that the main resonances are little affected by fields along the *b* direction of up to 0.6 Tesla.

FIG. 25. Schematic renormalization group flows for the two chain problem. It differs from previous proposals in that the flows in $t_\perp$ away from the decoupled chains fixed point may either flow into or approach closely a fixed point where there is finite, incoherent hopping between the chains. Which of those occurs depends on the stability of the incoherent fixed point to two body hopping between the chains and other perturbation generated by $t_\perp$.

FIG. 26. Schematic renormalization group flows for the many chain problem. It differs from previous proposals in that the flows in $t_\perp$ away from the decoupled chains fixed point may either flow into or approach closely a fixed point where there is finite, incoherent hopping between the chains. Experimentally, it appears that the incoherent fixed point is stable for $(TMTSF)_2PF_6$ in the presence of a magnetic field. It may be more generally stable, but in the case of $(TMTSF)_2PF_6$, it is unstable to superconductivity in zero magnetic field.